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ON PERIODIC SOLUTIONS OF RANDOM DIFFERENTIAL INCLUSIONS

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Abstract. By applying the random coincidence degree we develop the methods of random generalized smooth and nonsmooth integral guiding functions and use them for the study of periodic solutions for random differential inclusions in finite dimensional spaces.

1. Introduction

Let (Ω, Σ, μ) be a complete probability space (see, e.g., [6]) and $I = [0, T]$. In this paper we consider the periodic problem for a random differential inclusion of the form:

$$\begin{cases} x'(\omega, t) \in F\big(\omega, t, x(\omega, t)\big), & \text{for a.e. } t \in I, \\ x(\omega, 0) = x(\omega, T), \end{cases} \tag{1.1}$$

for all $\omega \in \Omega$, where $F\colon \Omega \times I \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a given multivalued map.

For the study of problem (1.1) we present the method of random generalized integral guiding functions. Let us mention that the method of guiding functions was developed by Krasnoselskii and Perov (see, e.g., [26, 27]) for the investigation of periodic oscillations in dynamical systems governed by differential equations. The notion of guiding function was then generalized in several directions and applied to various problems. Among a large number of papers on this subject let us recall: the paper of Mawhin [32] considered periodic solutions of functional differential equations; the work of Fonda [9], in which the notion of integral guiding functions was introduced; the paper of Rachinskii [38] in which defined the notion of multivalent guiding functions; the work of Górniewicz and Plaskacz [12, 13] in which the notion of general form of guiding functions for differential inclusions was presented (see also [5]); the paper of Lewicka [29] for nonsmooth guiding functions; a

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number of papers of Kornev, Loi, Obukhovskii, Zecca and Yao [15]- [24], [31] concerned with extension of the notion of guiding functions; the work of Loi [30] for the method of guiding functions in infinite dimensional Hilbert spaces; the papers of Kryszewski [28], Kryszewski and Gabor [10]; Loi, Obukhovskii and their coauthors [34]- [36], Kornev and Liou [25] for the application of the method of guiding functions to bifurcation problems. The backgrounds and applications of the method of guiding functions in nonlinear analysis can be found in the recent monograph [33].

Recently, Andres and Górniewicz [1] introduced the random topological degree and developed the method of random guiding functions for the study of random periodic solutions of random differential inclusions in finite dimensional spaces. In the present paper, based on the approach given in [1, 31] we present the notion of a random coincidence degree, the notions of smooth and nonsmooth random generalized integral guiding functions and use them to prove some existence theorems of random periodic solutions to problem (1.1).

The paper is organized in the following way. In the next section we recall some notions from theory of linear Fredholm operators, multivalued analysis and theory of random coincidence degree. The notions of random generalized integral guiding functions and of random nonsmooth generalized integral guiding functions are presented in Section 3 and 4, respectively. The main results are Theorems 3.6, 4.4 and 4.7.

2. Preliminaries and notation

2.1. **Fredholm operators.** Let X, Y be Banach spaces. At first, let us recall some notions from the theory of linear Fredholm operators (see, e.g., [11]).

A linear bounded operator $L\colon domL \subseteq X \to Y$ is said to be a *linear Fredholm operator of index zero* if

(i) $Im\, L$ is a closed subset of Y;
(ii) The spaces $Ker\, L$ and $Coker\, L$ are finite-dimensional and

$$dimKer\, L = dimCoker\, L.$$

For every linear Fredholm operator of zero index $L\colon domL \subseteq X \to Y$ there exist projections $P_L\colon X \to X$ and $Q_L\colon Y \to Y$ such that $Im\, P_L = KerL$ and $Ker\, Q_L = ImL$. If we define the operator

$$L_{P_L}\colon domL \cap KerP_L \to ImL$$

as the restriction of L on $domL \cap Ker\, P_L$, then L_{P_L} is a linear isomorphism and we can define the operator $K_{P_L}\colon ImL \to domL$ as $K_{P_L} = L_{P_L}^{-1}$. Now let $CokerL = Y/ImL$; $\Pi_L\colon Y \to CokerL$ be a canonical operator projection

$$\Pi_L(z) = z + ImL$$

and $\Lambda_L\colon CokerL \to KerL$ be a linear continuous isomorphism. Then the equation

$$Lx = y,\ y \in Y$$

is equivalent to the following one:

$$x = P_L x + (\Lambda_L \Pi_L + K_L)y,$$

where $K_L\colon Y \to X$ is defined as

$$K_L = K_{P_L}(i - Q_L).$$

2.2. **Multimaps and random coincidence degree.** We describe now some notions of the theory of multivalued maps that will be used in the sequel.

Let X, Y be metric spaces. Denote

$$P(Y) = \{\mathcal{M} \subset Y\colon \mathcal{M} \neq \emptyset\},$$
$$C(Y) = \{\mathcal{M} \in P(Y)\colon \mathcal{M} \text{ is closed}\},$$
$$K(Y) = \{\mathcal{M} \in P(Y)\colon \mathcal{M} \text{ is compact}\}.$$

Definition 2.1 (see, e.g., [5,13,14])**.** A multivalued map (multimap) $\mathcal{F}\colon X \to P(Y)$ is said to be:

- (i) *upper semicontinuous* (u.s.c) if $\mathcal{F}^{-1}(V) = \{y \in X\colon \mathcal{F}(y) \cap W \neq \emptyset\}$ is a closed subset of Y for every closed set $W \subset Y$;
- (ii) *lower semicontinuous* (l.s.c) if $\mathcal{F}^{-1}(V) = \{y \in X\colon \mathcal{F}(y) \cap V \neq \emptyset\}$ is an open subset of Y for every open set $V \subset Y$;
- (iii) *continuous* if it is both u.s.c. and l.s.c.;
- (iv) *closed* if its graph $\Gamma_{\mathcal{F}} = \{(y,z)\colon z \in \mathcal{F}(y)\}$ is a closed subset of $X \times Y$;
- (v) *compact* if the set $\overline{\mathcal{F}(X)}$ is compact in Y;
- (vi) *completely u.s.c.* if $\mathcal{F}$ is u.s.c. and the set $\mathcal{F}(U)$ is relatively compact in Y for each bounded set $U \subset X$;

Definition 2.2 (see [1])**.** Multimap $\mathcal{F}\colon \Omega \times X \to C(Y)$ is called a *random multioperator* if it is product-measurable (see, e.g., [6]), i.e. measurable w.r.t. $\Sigma \otimes \mathbb{B}(X)$, where $\Sigma \otimes \mathbb{B}(X)$ is the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times B$, where $A \in \Sigma$ and $B \in \mathbb{B}(X)$ and $\mathbb{B}(X)$ denotes the Borel σ-algebra on X. If, moreover, $\mathcal{F}(\omega, \cdot)\colon X \to C(Y)$ is u.s.c. for all $\omega \in \Omega$, then $\mathcal{F}$ is called a *random u-multioperator.*

Definition 2.3 (see [1])**.** Let $A \subset Y$ be a closed subset and $\mathcal{F}\colon \Omega \times A \to P(Y)$ a random multioperator. *A random fixed point* ξ of $\mathcal{F}$ is a measurable map $\xi\colon \Omega \to A$ such that

$$\xi(\omega) \in \mathcal{F}(\omega, \xi(\omega)),\ \forall \omega \in \Omega.$$

Theorem 2.4 (see [1])**.** *Let Y be a separable Banach space, $\mathcal{F}\colon \Omega \times Y \to C(Y)$ a random multioperator. If for each $\omega \in \Omega$ the set*

$$Fix\mathcal{F}_\omega := \{x \in Y\colon x \in \mathcal{F}(\omega, x)\}$$

of fixed points of $\mathcal{F}_\omega = \mathcal{F}(\omega, \cdot)$ is nonempty and closed then $\mathcal{F}$ has a random fixed point.

Definition 2.5. A multimap $\mathcal{F}\colon \Omega \times X \to K(Y)$ is said to be:

- $a)$ a *random compact u-multioperator* if it is a random u-multioperator and for each $\omega \in \Omega$ the multimap $\mathcal{F}(\omega, \cdot)\colon X \to K(Y)$ is compact;
- $b)$ a *random completely u-multioperator* if it is a random u-multioperator and for each $\omega \in \Omega$ the multimap $\mathcal{F}(\omega, \cdot)\colon X \to K(Y)$ is completely u.s.c.

Now, let Y be a separable Banach space, $Kv(Y)$ denote a collection of all nonempty compact convex subsets of Y, $U \subset Y$ an open bounded subset and $\mathcal{F}\colon \Omega \times \overline{U} \to Kv(Y)$ a random compact u-multioperator such that $x \notin \mathcal{F}(\omega, x)$ for all $x \in \partial U$ and for all $\omega \in \Omega$, where ∂U denotes the boundary of U. Then for each $\omega \in \Omega$ the topological degree of the corresponding multivalued vector field $deg\big(i - \mathcal{F}(\omega, \cdot), \overline{U}\big)$ is well defined (see, e.g., [2,5,13,14,28] . The random topological degree of $i - \mathcal{F}$ on $\overline{U}$ is defined as following (see [1]):

$$D(i - \mathcal{F}, \overline{U}) := \Big\{ deg\big(i - \mathcal{F}(\omega, \cdot), \overline{U}\big) \mid \omega \in \Omega \Big\}.$$

By $D(i - \mathcal{F}, \overline{U}) \neq 0$ we mean that $deg\big(i - \mathcal{F}(\omega, \cdot), \overline{U}\big) \neq 0$ for all $\omega \in \Omega$.

Theorem 2.6 (see [1])**.** *If $D(i - \mathcal{F}, \overline{U}) \neq 0$, then $\mathcal{F}$ has a random fixed point in U, i.e., there exists a measurable function $\xi\colon \Omega \to U$ such that $\xi(\omega) \in \mathcal{F}(\omega, \xi(\omega))$ for all $\omega \in \Omega$.*

Let X, Y be a separable Banach spaces, $Cv(Y)$ denote a collection of all nonempty closed convex subsets of Y, $U \subset X$ an open bounded subset, $L : \operatorname{dom} L \subseteq X \to Y$ a linear zero index Fredholm operator and $\mathcal{F} : \Omega \times \overline{U} \to Cv(Y)$ a random multioperator such that:

(*i*) $Lx \notin \mathcal{F}(\omega, x)$ for all $x \in \partial U \cap \operatorname{dom} L$ and $\omega \in \Omega$;
(*ii*) $(\Lambda_L \Pi_L + K_L) \circ \mathcal{F}(\omega, x)$ is a random compact u-multioperator.

Then for each $\omega \in \Omega$ the coincidence degree of the pair $(L, \mathcal{F}(\omega, \cdot))$ is defined as (see, e.g., [37], [39])

$$\deg(L, \mathcal{F}(\omega, \cdot), \overline{U}) := \deg(\Phi(\omega, \cdot), \overline{U}),$$

where

$$\Phi(\omega, x) = P_L x + (\Lambda_L \Pi_L + K_L) \circ \mathcal{F}(\omega, x).$$

The random coincidence degree of the pair $(L, \mathcal{F})$ is defined as

$$\operatorname{Deg}(L, \mathcal{F}, \overline{U}) := \{\deg(L, \mathcal{F}(\omega, \cdot), \overline{U}) \mid \omega \in \Omega\}.$$

We say that $\operatorname{Deg}(L, \mathcal{F}, \overline{U}) \neq 0$ provided $\operatorname{Deg}(L, \mathcal{F}(\omega, \cdot), \overline{U}) \neq 0$ for all $\omega \in \Omega$.

From the definition the next existence result easily follows.

Theorem 2.7. *If $\operatorname{Deg}(L, \mathcal{F}, \overline{U}) \neq 0$, then there exists a random coincidence point in U, i.e., there exists a measurable function $\xi\colon \Omega \to U \cap \operatorname{dom} L$ such that $L\xi(\omega) \in \mathcal{F}(\omega, \xi(\omega))$ for all $\omega \in \Omega$.*

2.3. **Notation.** For simplicity, we will use the same notation $|\cdot|$ $[\langle \cdot, \cdot \rangle]$ to denote the norm [resp., the inner product] in finite-dimensional spaces. Let I be a closed subset R endowed with the Lebesgue measure. By $C(I, \mathbb{R}^n)$ $[L^p(I, \mathbb{R}^n)$ $(p \geq 1)]$ we denote the spaces of all continuous [respectively, $p-$summable] functions $u\colon I \to \mathbb{R}^n$ with usual norms:

$$\|u\|_C = \max_{t \in I} |u(t)| \text{ and } \|u\|_p = \left(\int_0^T |u(t)|^p dt \right)^{\frac{1}{p}}.$$

Consider the space of all absolutely continuous functions $u\colon I \to \mathbb{R}^n$ whose derivatives belong to $L^p(I, \mathbb{R}^n)$. It is known (see, e.g., [3]) that this space can be identified with the Sobolev space $W^{1,p}(I, \mathbb{R}^n)$ with the norm

$$\|u\|_W = \Big(\|u\|_p^p + \|u'\|_p^p\Big)^{\frac{1}{p}}.$$

By $W_T^{1,p}(I, \mathbb{R}^n)$ we denote the space of all functions $x \in W^{1,p}(I, \mathbb{R}^n)$ satisfying the boundary condition of periodicity $x(0) = x(T)$. Recall that (see, e.g., [3]) the embedding $W^{1,2}(I, \mathbb{R}^n) \hookrightarrow C(I, \mathbb{R}^n)$ is compact. The symbols $B_C(0, r)$ $[B_{\mathbb{R}^n}(0, r)]$ denote the closed ball of radius r centered at 0 in the space $C(I, \mathbb{R}^n)$ [respectively, $\mathbb{R}^n$].

3. Random smooth generalized integral guiding functions

Now, let us consider problem (1.1). Assume that the following hypotheses hold true.

($F1$) $F\colon \Omega \times I \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ is a random u-multioperator;

($F2$) there exists $c > 0$ such that for every $\omega \in \Omega$:

$$\|F(\omega, t, y)\| := \sup\{|z| \colon z \in F(\omega, t, y)\} \le c(1 + |y|), \quad \forall y \in \mathbb{R}^n$$

for a.e. $t \in I$.

By a *random solution* of (1.1) we mean a function $\xi\colon \Omega \times I \to \mathbb{R}^n$ such that

1) the map $\omega \in \Omega \to \xi(\omega, \cdot) \in C(I, \mathbb{R}^n)$ is measurable:
2) for each $\omega \in \Omega$ the function $\xi(\omega, \cdot)$ is in $W^{1,2}(I, \mathbb{R}^n)$ and satisfies

$$\begin{cases} \xi'(\omega, t) \in F\big(\omega, t, \xi(\omega, t)\big), \\ \xi(\omega, 0) = \xi(\omega, T), \end{cases}$$

for a.e. $t \in I$.

From $(F1) - (F2)$ it follows that the superposition multioperator

$$\mathcal{P}_F\colon \Omega \times C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n)),$$
$$\mathcal{P}_F(\omega, x) = \{f \in L^2(I, \mathbb{R}^n) \colon f(s) \in F(\omega, s, x(s)), \quad \text{for a.e. } s \in I\},$$

is well defined. Moreover, for each $\omega \in \Omega$ the multimap $\mathcal{P}_F(\omega, \cdot)\colon C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n))$ is closed (see, e.g., [2, 5, 13, 14]).

Definition 3.1 (see [1, Definition 5.3])**.** A map $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is called a *random potential* if the following two conditions are satisfied:

(i) $V(\cdot, x)\colon \Omega \to \mathbb{R}$ is measurable for every $x \in \mathbb{R}^n$;

(ii) $V(\omega, \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ is a C^1-map for every $\omega \in \Omega$.

Definition 3.2 (see [1, Definition 5.4])**.** A random potential V is called a *random direct potential* if there exists $R_0 > 0$ such that

$$\nabla V(\omega, z) = \left(\frac{\partial V(\omega, z)}{\partial z_1}, \cdots, \frac{\partial V(\omega, z)}{\partial z_n}\right) \neq 0$$

for all $(\omega, z) \in \Omega \times \mathbb{R}^n \colon |z| \ge R_0$.

From the above definition it follows that for a fixed $\omega \in \Omega$ the topological degree

$$deg\big(\nabla V(\omega,\cdot), B_{\mathbb{R}^n}(0,R)\big)$$

is well-defined for all $R \geq R_0$ and it is nothing but $deg\big(\nabla V(\omega,\cdot), B_{\mathbb{R}^n}(0,R_0)\big)$.
By a *random index* $ind V$ of the random direct potential V we mean the random topological degree $D\big(\nabla V, B_{\mathbb{R}^n}(0,R_0)\big)$.

Definition 3.3. A random potential $V\colon \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ is said to be a *random strict integral guiding function* for problem (1.1) if there exists $N > 0$ such that for all $\omega \in \Omega$ from $x \in C(I,\mathbb{R}^n)$ with $\|x\|_2 \geq N$ it follows that

$$\int_0^T \big\langle \nabla V(\omega, x(s)), f(s)\big\rangle dt > 0 \quad \forall f \in \mathcal{P}_F(\omega, x). \tag{3.1}$$

It is easy to verify the following assertion.

Lemma 3.4. *If* $V\colon \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ *is a random strict integral guiding function for problem* (1.1)*, then it is a random direct potential, and hence there exists its random index* $indV$.

Definition 3.5. A random direct potential $V\colon \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ is said to be a *random generalized integral guiding function* for problem (1.1) if there exists $N > 0$ such that for all $\omega \in \Omega$ from $x \in C(I,\mathbb{R}^n)$ with $\|x\|_2 \geq N$ it follows that

$$\int_0^T \big\langle \nabla V(\omega, x(s)), f(s)\big\rangle dt \geq 0 \quad \text{for some } \ f \in \mathcal{P}_F(\omega, x). \tag{3.2}$$

Now we are in position to prove the main result of this section.

Theorem 3.6. *Let conditions* $(F1)-(F2)$ *hold. If there exists a random generalized integral guiding function* V *for problem* (1.1) *such that* $indV \neq 0$*, then problem* (1.1) *has a random solution.*

Proof. **Step 1.** Let us consider the case of the random strict integral guiding function for problem (1.1). Define the operator $L\colon W_T^{1,2}(I,\mathbb{R}^n) \to L^2(I,\mathbb{R}^n)$, $Lx = x'$. It is well known (see, e.g., [11]) that L is a linear Fredholm operator of index zero and

$$Ker\, L \cong \mathbb{R}^n \cong Coker\, L.$$

The projection

$$\Pi_L\colon L^2(I,\mathbb{R}^n) \to \mathbb{R}^n,$$

is defined as

$$\Pi_L g = \frac{1}{T}\int_0^T g(s)\,ds$$

and the homeomorphism $\Lambda_L\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator. The space $L^2(I,\mathbb{R}^n)$ can be represented as

$$L^2(I,\mathbb{R}^n) = \mathcal{L}_0 \oplus \mathcal{L}_1,$$

where $\mathcal{L}_0 = Coker\, L$ and $\mathcal{L}_1 = Im\, L$.
The decomposition of an element $g \in L^2(I,\mathbb{R}^n)$ is denoted by

$$g = g^{(0)} + g^{(1)}, g^{(0)} \in \mathcal{L}_0, g^{(1)} \in \mathcal{L}_1.$$

Problem (1.1) can be substituted with the following operator inclusion

$$Lx \in \mathcal{P}_F(\omega, x),$$

or equivalently, by the fixed point problem

$$x \in \Gamma(\omega, x), \tag{3.3}$$

where the multimap $\Gamma\colon \Omega \times C(I, \mathbb{R}^n) \multimap C(I, \mathbb{R}^n)$ is defined as

$$\Gamma(\omega, x) = P_L x + (\Pi_L + K_L) \circ \mathcal{P}_F(\omega, x).$$

Let us show that Γ is measurable. In fact, it is sufficient to prove that $\mathcal{P}_F$ is measurable. To do this, notice that from $(F1)$ it follows that for a given $(\omega, x) \in \Omega \times C(I, \mathbb{R}^n)$ the multifunction $t \in I \multimap F(\omega, t, x(t))$ is measurable. Now, let us fix an arbitrary $g \in L^2(I, \mathbb{R}^n)$ and define the function

$$h_g\colon \Omega \times C(I, \mathbb{R}^n) \to [0, \infty),$$
$$h_g(\omega, x) = dist_{L^2(I,\mathbb{R}^n)}\big(g, \mathcal{P}_F(\omega, x)\big).$$

Applying [8], Proposition 3.4 (b) we conclude that

$$h_g(\omega, x) = \Big(\int_0^T dist^2_{\mathbb{R}^n}(g(s), F(\omega, s, x(s))ds\Big)^{1/2}.$$

From the Fubini theorem it follows that the map h_g is measurable and hence (see, e.g., [6], Ch.III or [5], Theorem 1.5.6) the multimap $\mathcal{P}_F$ is measurable.

Now, for every $\omega \in \Omega$ let us prove that $\Gamma(\omega, \cdot)\colon C(I, \mathbb{R}^n) \multimap C(I, \mathbb{R}^n)$ is a completely u.s.c. multimap with compact, convex values. Indeed, from the fact that the operator $\Pi_L + K_L$ is linear and continuous it follows that the multimap $(\Pi_L + K_L) \circ \mathcal{P}_F(\omega, \cdot)$ is closed (see, e.g., Theorem 1.5.30 [5] or Corollary 5.1.2 [14]). Further, from $(F2)$ it follows that for every bounded subset $U \subset C(I, \mathbb{R}^n)$ the set $(\Pi_L + K_L) \circ \mathcal{P}_F(\omega, U)$ is bounded in $W^{1,2}_T(I, \mathbb{R}^n)$, and by the Sobolev embedding theorem (see, e.g., [3]) it is a relatively compact subset of $C(I, \mathbb{R}^n)$. Therefore, the multimap $(\Pi_L + K_L) \circ \mathcal{P}_F(\omega, \cdot)$ is u.s.c. and now the assertion follows from the fact that P_L is continuous and has a finite-dimensional range. So, Γ is a random completely u-multioperator.

Now our aim is to evaluate the coincidence degree of the pair (L, Γ) on a ball of a sufficiently large radius. Fix $\omega \in \Omega$ and assume that $x_\omega \in C(I, \mathbb{R}^n)$ is a solution to the inclusion (3.3). Then there exist $f_\omega \in \mathcal{P}_F(\omega, x_\omega)$ such that

$$\begin{cases} x'_\omega(t) = f_\omega(t), \text{ for a.e. } t \in I, \\ x_\omega(0) = x_\omega(T). \end{cases}$$

Therefore,

$$\int_0^T \big\langle \nabla V(\omega, x_\omega(t)), f_\omega(t)\big\rangle dt = \int_0^T \big\langle \nabla V(\omega, x_\omega(t)), x'_\omega(t)\big\rangle dt = 0.$$

Consequently, $\|x_\omega\|_2 < N$. From $(F2)$ it follows that there exists $M > 0$ such that $\|x'_\omega\|_2 < M$. So, we can choose $R_1 > 0$ which does not depend on ω such that $\|x_\omega\|_C < R_1$.

Let $R = R_1 + 1$. Then for any $\omega \in \Omega$ inclusion (3.3) has only trivial solutions on $B_C(0,R)$. Therefore, the degree $deg(L, \Gamma(\omega,\cdot), B_C(0,R))$ is well-defined for every $\omega \in \Omega$. To evaluate this characteristic, we consider the following family of multimaps

$$\Psi_\omega \colon B_C(0,R) \times [0,1] \to Kv(C(I,\mathbb{R}^n)),$$

$$\Psi_\omega(x,\eta) = P_L x + (\Pi_L + K_L) \circ \varphi\big(\mathcal{P}_F(\omega,x),\eta\big),$$

where the map $\varphi \colon L^2(I,\mathbb{R}^n) \times [0,1] \to L^2(I,\mathbb{R}^n)$ is defined as

$$\varphi(g,\eta) = g^{(0)} + \eta g^{(1)}, \tag{3.4}$$

when $g^{(0)} \in \mathcal{L}_0$, $g^{(1)} \in \mathcal{L}_1$ and $g = g^{(0)} + g^{(1)}$.
It is easy to verify that Ψ is a compact u.s.c. multimap. Let us show that

$$x \notin \Psi_\omega(x,\eta)$$

for all $(x,\eta) \in \partial B_C(0,R) \times [0,1]$. To the contrary, assume that there is $(x_*,\eta_*) \in \partial B_C(0,R) \times [0,1]$ such that $x_* \in \Psi_\omega(x_*,\eta_*)$. Then there exist $f_* \in \mathcal{P}_F(\omega,x_*)$ such that

$$\begin{cases} x_*'(t) = \varphi\big(f_*,\eta_*\big)(t) \text{ for a.e. } t \in I, \\ x_*(0) = x_*(T), \end{cases}$$

or equivalently,

$$\begin{cases} x_*' = \eta_* f_*^{(1)}, \\ 0 = f_*^{(0)}, \end{cases}$$

where $f_*^{(0)} + f_*^{(1)} = f_*$, $f_*^{(0)} \in \mathcal{L}_0$, $f_*^{(1)} \in \mathcal{L}_1$.
If $\eta_* \neq 0$, then

$$\int_0^T \big\langle \nabla V(\omega, x_*(t)), f_*(t) \big\rangle dt = \frac{1}{\eta_*} \int_0^T \big\langle \nabla V(\omega, x_*(t)), x_*'(t) \big\rangle dt = 0.$$

Therefore, $\|x_*\|_2 < N$, and hence $\|x_*\|_C \leq R_1 < R$, giving a contradiction.
If $\eta_* = 0$, then $x_* \in Ker\, L$, i.e., $x_*(t) \equiv z \in \mathbb{R}^n$ for all $t \in I$. Since $\|z\|_2 > N$ we have

$$\int_0^T \big\langle \nabla V(\omega, z), \gamma(t) \big\rangle dt = T \big\langle \nabla V(\omega,z), \Pi_L \gamma \big\rangle > 0, \tag{3.5}$$

for all $\gamma \in \mathcal{P}_F(\omega, z)$.
In particular,

$$0 < \big\langle \nabla V(\omega,z), \Pi_L f_* \big\rangle = \big\langle \nabla V(\omega,z), \Pi_L f_*^{(0)} \big\rangle = 0,$$

that is the contradiction.
Thus, Ψ_ω is a homotopy connecting the multimaps $\Psi_\omega(\cdot,1) = \Gamma(\omega,\cdot)$ and

$$\Psi_\omega(\cdot,0) = P_L + \Pi_L \circ \mathcal{P}_F(\omega,\cdot).$$

By virtue of the homotopy invariance property of the topological degree we have

$$deg(i - \Gamma(\omega,\cdot), B_C(0,R)) = deg(i - P_L - \Pi_L \mathcal{P}_F(\omega,\cdot), B_C(0,R)).$$

Notice that the multimap $P_L + \Pi_L \mathcal{P}_F(\omega,\cdot)$ takes values in $\mathbb{R}^n$, and hence, by the Map Restriction Property (see, e.g., [5,14]):

$$deg(i - P_L - \Pi_L \mathcal{P}_F(\omega,\cdot), B_C(0,R)) = deg(i - P_L - \Pi_L \mathcal{P}_F(\omega,\cdot), B_{\mathbb{R}^n}(0,R)).$$

In the space $\mathbb{R}^n$ the vector multifield $i - P_L - \Pi_L \mathcal{P}_F(\omega, \cdot)$ has the form:

$$i - P_L - \Pi_L \mathcal{P}_F(\omega, \cdot) = -\Pi_L \mathcal{P}_F(\omega, \cdot).$$

From (3.5) it easily follows that $-\Pi_L \mathcal{P}_F(\omega, \cdot)$ and $-\nabla V(\omega, \cdot)$ are homotopic on $\partial B_{\mathbb{R}^n}(0, R)$. So we obtain

$$deg(i - \Gamma(\omega, \cdot), B_C(0, R)) = deg(-\Pi_L \mathcal{P}_F(\omega, \cdot), B_{\mathbb{R}^n}(0, R)) = \\ = (-1)^n deg\big(\nabla V(\omega, \cdot), B_{\mathbb{R}^n}(0, R)\big) \neq 0.$$

Therefore, the random coincidence degree $\mathrm{Deg}\,(L, \Gamma, B_C(0, R)) \neq 0$. Applying Theorem 2.7 we obtain that problem (1.1) has a random solution.

Step 2. Now we consider the case of the random generalized integral guiding function for problem (1.1).

Consider a multimap $B : C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n))$ defined as

$$B(x) = \left\{ \varphi : \ |\varphi(t)| \leq c(1 + \|x_t\|) \text{ and } \gamma(x) \int_0^T \big\langle \nabla V\big(x(s)\big), \varphi(s) \big\rangle \, ds \geq 0 \right\},$$

for a.e. $t \in [0, T]$, c is a constant from the condition $(F2)$,

$$\gamma(x) = \begin{cases} 0, & \text{if } \|x\|_2 \leq N, \\ 1, & \text{if } \|x\|_2 > N. \end{cases}$$

It is clear that B is a closed multimap.

Let us consider a multimap $\mathcal{P}_F^B \colon \Omega \times C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n))$ given as

$$\mathcal{P}_F^B(\omega, x) = \mathcal{P}_F(\omega, x) \cap B(x).$$

Obviously, for each $\omega \in \Omega$ the multimap $\mathcal{P}_F^B(\omega, \cdot) \colon C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n))$ is closed and the condition (3.2) is satisfied for all $f \in \mathcal{P}_F^B(\omega, x)$.

For the random direct potential V we define a map $Y_V : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$Y_V(\omega, x) = \begin{cases} \nabla V(\omega, x), & \text{if } \|\nabla V(\omega, x)\| \leq 1, \\ \frac{\nabla V(\omega, x)}{\|\nabla V(\omega, x)\|}, & \text{if } \|\nabla V(\omega, x)\| > 1. \end{cases}$$

It is easy to see that the map Y is continuous.

For any $\varepsilon_m > 0$ we define a multimap $\mathcal{P}_F^m \colon \Omega \times C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n))$ as following

$$\mathcal{P}_F^m(\omega, x) = \mathcal{P}_F^B(\omega, x) + \varepsilon_m Y_V(\omega, x).$$

It is clear for each $\omega \in \Omega$ the multimap $\mathcal{P}_F^m(\omega, \cdot) \colon C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n))$ is closed and for each $\varepsilon_m > 0$ the condition (3.1) is fulfilled for all $f \in \mathcal{P}_F^m(\omega, x)$. By applying results of Step 1 we can prove for each $\varepsilon_m > 0$ the solvability of the following operator inclusion

$$Lx \in \mathcal{P}_F^m(\omega, x),$$

or, equivalently, the existence of a fixed point

$$x_m \in P_L x_m + (\Lambda_L \Pi_L + K_L) \circ \mathcal{P}_F^m(x_m),$$

from where the existence of a solution for problem (1.1) follows. □

4. Random nonsmooth generalized integral guiding functions

Firstly, let us recall some notions of non-smooth analysis (see, e.g., [7]). Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz functional. For every $y_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$ *the generalized directional derivative* $V^0(y_0;\nu)$ of V at the point y_0 in the direction ν is defined as

$$V^0(y_0;\nu) = \overline{\lim_{\substack{y \to y_0 \\ t \downarrow 0}}} \frac{V(y+t\nu) - V(y)}{t}. \tag{4.1}$$

By Proposition 2.1.1 [7], the functional $V^0\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is upper semicontinuous, i.e., for each sequences $(y_n,\nu_n) \in \mathbb{R}^n \times \mathbb{R}^n$, $(y_n,\nu_n) \to (y_0,\nu_0)$, the following relation holds:

$$\overline{\lim_{n\to+\infty}} V^0(y_n;\nu_n) \leq V^0(y_0;\nu_0).$$

The generalized gradient $\partial V(x_0)$ of functional V at $y_0 \in \mathbb{R}^n$ is defined by:

$$\partial V(y_0) = \left\{ y \in \mathbb{R}^n \colon \langle y,\nu\rangle \leq V^0(y_0;\nu) \text{ for every } \nu \in \mathbb{R}^n \right\}.$$

It is well known (see, e.g., [7]) that the multimap $\partial V\colon \mathbb{R}^n \to P(\mathbb{R}^n)$ is u.s.c. and has compact convex values. In particular, it means that for every continuous function $x\colon [0,T] \to \mathbb{R}^n$ the set $\mathcal{P}_{\partial V}(x)$ of all summable selections of the multifunction $\partial V(x(t))$ is non-empty.

A locally Lipschitz functional $V\colon \mathbb{R}^n \to \mathbb{R}$ is called *regular,* if for every $y \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$ there exists the directional derivative $V'(y,\nu)$ and $V'(y,\nu) = V^0(y,\nu)$. It is known (see, e.g., [7]) that locally bounded convex functionals are regular.

Lemma 4.1 (see [20]). *Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a regular functional, $x\colon [0,T] \to \mathbb{R}^n$ an absolutely continuous function. Then the function $V(x(t))$ is absolutely continuous and*

$$V(x(t)) - V(x(0)) = \int_0^t V^0(x(s),x'(s))ds, \; t \in [0,T].$$

Definition 4.2. A map $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is called a *random nonsmooth potential* if the following two conditions are satisfied:

- (i) $V(\cdot,x)\colon \Omega \to \mathbb{R}$ is measurable for every $x \in \mathbb{R}^n$;
- (ii) $V(\omega,\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz functional for every $\omega \in \Omega$.

Definition 4.3. A random nonsmooth potential $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *random nonsmooth strict integral guiding function* for problem (1.1), if the following conditions hold:

- (i) the function $V(\omega,\cdot)$ is regular for every $\omega \in \Omega$;
- (ii) there exists $N > 0$ such that for all $\omega \in \Omega$ from $x \in C(I,\mathbb{R}^n)$ with $\|x\|_2 \geq N$, it follows that

$$\int_0^T \langle \upsilon(t), f(t)\rangle dt > 0$$

for all $\upsilon \in \mathcal{P}_{\partial V}(\omega,x)$ and for all $f \in \mathcal{P}_F(\omega,x)$, where

$$\mathcal{P}_{\partial V}(\omega,x) = \left\{ \upsilon \in L^2(I,\mathbb{R}^n) \colon \upsilon(t) \in \partial V(\omega,x(t)) \text{ for a.e. } t \in I \right\}.$$

It is easy to verify that if V is the random nonsmooth strict integral guiding function for (1.1), then for every $\omega \in \Omega$ the topological degree $deg\big(\partial V(\omega,\cdot), B_{\mathbb{R}^n}(0,r)\big)$ of the multivalued vector field is well-defined for all $r \geq \frac{N}{\sqrt{T}}$. Denote $ind\, V = D\big(\partial V, B_{\mathbb{R}^n}(0,r)\big)$.

Analogously to Theorem 3.6 we obtain the following result.

Theorem 4.4. *Let conditions* $(F1) - (F2)$ *hold. If there exists a regular random nonsmooth strict integral guiding function for problem* (1.1) *such that* $ind\, V \neq 0$, *then problem* (1.1) *has a random solution.*

Definition 4.5. A random nonsmooth potential $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *random nonsmooth direct potential* if there exists $R_0 > 0$ such that

$$0 \notin \partial V(\omega, x)$$

for all $(\omega, z) \in \Omega \times \mathbb{R}^n\colon |z| \geq R_0$.

From the above definition it follows that for a fixed $\omega \in \Omega$ the topological degree $deg\big(\partial V(\omega,\cdot), B_{\mathbb{R}^n}(0,R)\big)$ is well-defined for all $R \geq R_0$ and it is nothing but $deg\big(\partial V(\omega,\cdot), B_{\mathbb{R}^n}(0,R_0)\big)$.

Definition 4.6. A random nonsmooth direct potential $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *random nonsmooth generalized integral guiding function* for problem (1.1), if except the condition (i) the following estimate holds:

$(ii)'$ there exists $N > 0$ such that for all $\omega \in \Omega$ from $x \in C(I, \mathbb{R}^n)$ with $\|x\|_2 \geq N$, it follows that

$$\int_0^T \langle \upsilon(t), f(t)\rangle dt \geq 0$$

for all $\upsilon \in \mathcal{P}_{\partial V}(\omega, x)$ and for all $f \in \mathcal{P}_F(\omega, x)$.

Theorem 4.7. *Let conditions* $(F1) - (F2)$ *hold. If there exists a regular random nonsmooth generalized integral guiding function for problem* (1.1) *such that* $ind\, V \neq 0$, *then problem* (1.1) *has a random solution.*

Proof. For $k \in N$ let us consider

$$M^k = \sup\left\{\, \|\mathcal{P}_{\partial V}(\omega, x)\| :\ \omega \in \Omega,\ x \in \overline{B}^n(k) \right\},$$

where $\overline{B}^n(k)$ denotes the closed ball radius k centered at 0. Following [4], we define a map $\eta : \mathbb{R}^n \to \mathbb{R}$ as

$$\eta(x) = 1 + (\|x\| - k)\, M^{k+2} + (k + 1 - \|x\|)\, M^{k+1}, \quad k \leq \|x\| \leq k+1.$$

It is clear, that the map η is continuous and the following condition holds

$$\eta(x) \geq \max\{1, \|\mathcal{P}_{\partial V}(\omega, x)\|\} \quad \text{for all } \omega \in \Omega,\ x \in \mathbb{R}^n.$$

A multimap $Y : \Omega \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ given as

$$Y(\omega, x) = \frac{\partial V(\omega, x)}{\eta(x)}$$

is random u-multioperator and satisfies the estimate $\|Y(\omega, x)\| \leq 1$ for all $\omega \in \Omega,\ x \in \mathbb{R}^n$.

We consider now an auxiliary periodic problem for a random differential inclusion of the following form

$$\begin{cases} x'(\omega,t) \in F_Y(\omega,t,x(\omega,t)) = F(\omega,t,x(\omega,t)) + \varepsilon_m Y(\omega,x), \\ x(\omega,0) = x(\omega,T). \end{cases} \tag{4.2}$$

for all $\omega \in \Omega$.

It easy to see that the function V is a random nonsmooth strict integral guiding function for problem (4.2). Thus the conditions of the theorem 4.4 are satisfied and the problem (4.2) has a random solution. It follows that the problem (1.1) has a random solution. □

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