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THE ASYMPTOTIC CONTENT OF SOLUTIONS ON SPACES WITH SINGULARITIES

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Abstract. This exposition is devoted to structures induced by interface and other singular configurations in applications in terms of Mellin quantizations and operator-valued symbols which reflect specific phenomena in a framework which controls solutions of problems in partial differential equations up to the strata of the underlying spaces.

1. Introduction

This paper gives a survey on recent new ideas in the analysis of partial differential equations (PDEs) on spaces M where the C^∞ structure is violated in a neighbourhood of some "singular" subset N. Such a situation is quite common in many applications, and there have been developed numerous approaches. It turns out that specific features of applications correspond to large fields of research activities. The gap between concrete practical aspects and structure insight of reasonable generality can be enormous, and, as is well-known, even parts of information can open voluminous considerations, cf. [5], [6], [12], [13], [20], [27]. Here we focus on ideas and methods from analysis on singular manifolds of some controlled degree k of singularities. The case $k = 0$ corresponds to smoothness in the standard sense, $k = 1$ to (regular) conical or edge singularities, etc. We outline here essential structures which may appear to be rather far from other "established" approaches, though general intentions are shared by different schools of recent research, cf. [21], [23].

2. Motivation of spaces with singularities

Intuitively we can imagine a space M with (in our case) geometric singularities as an extension of the concept of topological spaces which contains a subset N of singularities such that every $p \in M$ has a neighbourhood $V(p)$ in M which can be locally described by a singular chart when $p \in N$, i.e., a set which is roughly speaking a standard corner set, iteratively characterized by repeatedly forming cones, wedges, combined by subsequent globalizations, and then followed by the cone or wedge construction again. We postpone for the moment the discussion of the transition behaviour of such charts, but typical examples of singular charts are cones

$$X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) \tag{2.1}$$

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for any smooth compact manifold X or wedges

$$X^{\Delta} \times \mathbb{R}^q. \tag{2.2}$$

The novelties refer to any neighbourhood of the singular subset. In analytic descriptions we pass to open stretched cones

$$X^{\wedge} = \mathbb{R}_+ \times X \tag{2.3}$$

in "some" choice of splitting of variables $(r, x), r \in \mathbb{R}_+, x \in X$. Analogously we look at open stretched wedges

$$X^{\wedge} \times \mathbb{R}^q$$

in the splitting of variables (r, x, y), in obvious meaning of notation. In such simple cases it is easy to imagine the nature of transition maps. For instance, two different splittings of variables $(r, x), (\tilde{r}, \tilde{x})$ belong to the same cone structure if the transition from (r, x) to $(\tilde{r}, \tilde{x})$ is the restriction of a smooth map between the involved closed cylinders $\overline{\mathbb{R}}_+ \times X \to \overline{\mathbb{R}}_+ \times X$ to $\mathbb{R}_+ \times X$. Thus we can introduce an infinite (overcountable) variety of non-equivalent cone configurations, but each of those opens an equivalence class of cone structures. Globally we then obtain manifolds with conical singularities. The idea is even visible for $\dim X = 0$, i.e., for $\mathbb{R}_+$ which is regarded as a specific (open) cone. In a similar manner we can characterize wedge charts, with a natural notion of equivalence of splittings of variables (r, x, y) and $(\tilde{r}, \tilde{x}, \tilde{y})$, respectively, which gives us manifolds with edges. This allows us to continue the construction, and by repeatedly (k times) carrying out the process we reach

$$\mathfrak{M}_k,$$

the category of spaces of singularity order k.

Let us now look at a few examples, namely $M \in \mathfrak{M}_1$ which can be a "global" manifold with conical singularities or with edges. Replacing the former X by M we obtain in an obvious manner spaces in $\mathfrak{M}_2$, etc. Even for this case the respective spaces may be "non-elementary". For instance, if $X_1, X_2 \in \mathfrak{M}_0$ are compact smooth manifolds, the spaces $X_1^{\Delta}, X_2^{\Delta}$ belong to $\mathfrak{M}_1$ and we have

$$X_1^{\Delta} \times X_2^{\Delta} \in \mathfrak{M}_2.$$

It is not easy to imagine the corner geometry of this space, though this belongs to the harmless examples.

3. The analysis induced by non-smooth geometries

In cases of non-smooth geometries we follow a similar scheme as in the smooth case where there is only one kind of charts which map a coordinate neighbourhood $V \subset M$ into an Euclidean space $\mathbb{R}^n$ for $n = \dim M$.

Already on a manifold M with smooth boundary ∂M we have the former situation over $M \setminus \partial M$, but close to ∂M the charts map to half-spaces $\overline{\mathbb{R}}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. In this way such spaces M belong to $\mathfrak{M}_1$.

Concerning $M \in \mathfrak{M}_k$ we have $k+1$ classes of charts and every $p \in M$ is contained in one of them. For any p we can determine a minimal $l, 0 \leq l \leq k$, and interpret the above neighbourhood $V(p)$ as an element of $\mathfrak{M}_l$. Thus appart from global effects in

PDE-problems we carry out local (singular) operations. This discussion is aimed at understanding the analysis on singular spaces, and we intend to explain the nature of substitutes of the Fourier transform of standard distribution spaces and adequate analogues of pseudo-differential operators who will be the adequate condidates of solving equations, i.e., of constructing parametrices to elliptic PDE.

4. Example: Reduction of boundary conditions to the boundary

In order to illustrate important cases where singularities of a space are realized as interfaces and where the geometry near a boundary induces in a natural manner pseudo-differential operators we outline some ideas around the principle of reducing a boundary condition to the boundary. We begin with the case of the Dirichlet problem for the standard Laplacian Δ in a smooth bounded domain Ω in $\mathbb{R}^n$, regarded as a continuous operator

$$\mathcal{A}_0 := \begin{pmatrix} \Delta \\ T_0 \end{pmatrix} : C^\infty(\overline{\Omega}) \to \begin{matrix} C^\infty(\overline{\Omega}) \\ \oplus \\ C^\infty(\partial\Omega) \end{matrix}. \tag{4.1}$$

For simplicity for the moment we realize the operator between spaces of smooth functions, where

$$T_0 u := u|_{\partial\Omega}$$

is the restriction operator which corresponds to Dirichlet conditions for u. We employ the well-known fact that $\mathcal{A}_0$ determines an isomorphism, and the inverse will be written as a row matrix

$$\mathcal{P}_0 := \begin{pmatrix} P_0 & K_0 \end{pmatrix} : \begin{matrix} C^\infty(\overline{\Omega}) \\ \oplus \\ C^\infty(\partial\Omega) \end{matrix} \to C^\infty(\overline{\Omega}). \tag{4.2}$$

The operator P_0 is of the form $P_0 = E + G_0$ for a fundamental solution E of the Laplacian and a so-called Green operator G_0. The operator P_0 is also called Green's function of the Dirichlet problem, and K_0 is also called the double layer potential. The identity

$$\mathcal{A}_0\mathcal{P}_0 := \begin{pmatrix} \Delta P_0 & \Delta K_0 \\ T_0 P_0 & T_0 K_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

allows us reduce the Neumann problem

$$\mathcal{A}_1 := \begin{pmatrix} \Delta \\ T_1 \end{pmatrix} : C^\infty(\overline{\Omega}) \to \begin{matrix} C^\infty(\overline{\Omega}) \\ \oplus \\ C^\infty(\partial\Omega) \end{matrix} \tag{4.4}$$

to the boundary by means of $\mathcal{A}_0$, according to

$$\mathcal{A}_1\mathcal{P}_0 := \begin{pmatrix} \Delta P_0 & \Delta K_0 \\ T_1 P_0 & T_1 K_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T_1 P_0 & T_1 K_0 \end{pmatrix} \tag{4.5}$$

for $T_1 u = \frac{\partial}{\partial\nu} u|_{\partial\Omega}$ with $\frac{\partial}{\partial\nu}$ being the derivative in inner normal direction. The operator $R := T_1 K_0$ is a first order classical pseudo-differential operator on the boundary and just represents the reduction of the Neumann condition T_1 to the boundary. This process has been analyzed and interpreted in the first decades of developing the pseudo-differential analysis which is a natural way to generate pseudo-differential operators purely in terms of standard techniques using differential operators, see Hörmanders work [15]. Originally Hörmander applied some elementary computations. Here we tacitly took material from Boutet-de Monvel's calculus [2], which was developed to formulating an operator algebra for boundary value problems with compositions as soon as rows and columns fit together in the middle. The original purpose of [2] was also to prove an analogue of the Atiyah-Singer-Index Theorem on manifolds with boundary.

Other examples of reducing elliptic boundary conditions to the boundary are explicitly elaborated in the monograph [11] jointly with Harutynyan. In this case the resulting reduced operators R are Douglis-Nirenberg elliptic and may have a non-vanishing Fredholm index as operators in Sobolev spaces. Then, relation (4.5) gives an index identity, namely,

$$\operatorname{ind}\mathcal{A}_1 - \operatorname{ind}\mathcal{A}_0 = \operatorname{ind} R \tag{4.6}$$

(in this case the summands all vanish), the meaning of (4.6) is the well-known Agranovich-Dynin formula and compares the indices of two elliptic boundary value problems for the same elliptic operator. Before we come to further questions which also concern interface problems we draw some conclusions in the simplest case of reduction to the boundary which we sketched so far. Setting for the moment

$$N := T_1 P_0$$

it is instructive to see that relation (4.5) which takes the form

$$\mathcal{A}_1\mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ N & R \end{pmatrix} \tag{4.7}$$

allows us to express a parametrix

$$\mathcal{P}_1 := \begin{pmatrix} P_1 & K_1 \end{pmatrix} \tag{4.8}$$

for the operator $\mathcal{A}_1$. In fact, we have

$$(\mathcal{A}_1\mathcal{P}_0)^{(-1)} \sim \mathcal{A}_0\mathcal{P}_1 \sim \begin{pmatrix} 1 & 0 \\ -R^{(-1)}N & R^{(-1)} \end{pmatrix} \tag{4.9}$$

with upper (-1) indicating a parametrix of the respective operator. Here we employ the fact that parametrices exist in Boutet de Monvel's calculus and "$\sim$" means equivalence modulo smoothing operators in that calculus. Thus, using notation in (4.2) and freely composing operators in Boutet de Monvel's set-up we can write

$$\mathcal{P}_1 \sim \mathcal{P}_0 \begin{pmatrix} 1 & 0 \\ -R^{(-1)}N & R^{(-1)} \end{pmatrix} \sim \begin{pmatrix} P_0 - K_0R^{(-1)}N & K_0R^{(-1)} \end{pmatrix} \tag{4.10}$$

i.e., it follows that

$$\mathcal{P}_1 = \begin{pmatrix} P_0 + G_1 & K_1 \end{pmatrix} \tag{4.11}$$

for

$$G_1 = -K_0R^{(-1)}N, \;\; K_1 = K_0R^{(-1)},$$

i.e., $\mathcal{P}_1$ is expressed in terms of $\mathcal{A}_0$ and $R^{(-1)}$.

Remark 4.1. *The operators G_1 are just of Green type while K_0, K_1 are operators of potential type in Boutet de Monvel's calculus.*

In a next step we pass to interpreting the Zaremba problem in connection with singular interface problems. Here we partly refer to joint work with Habal [4].

The Zaremba problem in its simplest form is formulated for the Laplacian Δ in a smooth domain where the boundary $\partial\Omega$ is subdivided into submanifolds Y_- and Y_+ with common boundary $Z = Y_- \cap Y_+$, first assumed to be smooth and of codimension 1 on $\partial\Omega$, such that $\partial\Omega = Y_- \cup Y_+$, and where we pose Dirichlet conditions on $\operatorname{int} Y_-$ and Neumann conditions on $\operatorname{int} Y_+$. In other words the Zaremba problem represents a boundary value problem where the conditions have a jump along the interface Z. As is well known this situation is much more difficult than the above mentioned boundary value problems. The jump of the conditions causes that the Fredholm property of associated operators is violated, and a careful analysis shows that similarly as boundary conditions we have to impose extra interface conditions along Z of trace and potential type. Thus, in contrast to Boutet de Monvel's calculus with its 2×2 block matrix operators we now have 3×3 matrices with trace, potential, Green operators mapping data over Z to distributions on Y_-, Y_+, Z as well as in converse direction. In addition we see that the transmission property is violated across Z on the boundary. In models of physics the situation also makes sense for parabolic equations with jumping conditions along interfaces on a spatial-time cylinder. In any case it is interesting to admit also interfaces Z which have a non-smooth geometry, for instance conical points or edges. However, if Z itself is smooth then, if we return to the Zaremba problem in above-mentioned form, the reduction of Zaremba boundary conditions, represented by $T_0|_{\operatorname{int} Y_-}, T_1|_{\operatorname{int} Y_+}$ to the boundary yields a pair of operators $(1, R_+)$ where 1 is the identity operator on the Dirichlet side, R_+ the restriction to $\operatorname{int} Y_+$ of the above operator R to the Neumann side. There is a process, elaborated in joint work with Chang and Habal, cf. [4], to perform the reduction back, i.e., to reproduce the original Zaremba problem from the operator R_+ on the Neumann side, and also to associate a Fredholm 3×3 block matrix operator of the transmission algebra in one-to-one correspondence with a 2×2 block matrix operator on Y_+ with R_+ in the upper left corner together with Green, potential and trace operators, as is known from the edge pseudo-differential algebra with Y_+ being treated as a manifold with edge Z, though in this case with a one-dimensional model cone $\overline{\mathbb{R}}_+$, as in boundary value problems. Nevertheless the operator R_+ violates the transmission property in a spectacular way, and in order to treat the problem we have to "switch on" the tools of the pseudo-differential edge calculus, cf. [9], [26], [27] or [29]. In this way, we illustrated that a quite obvious physical situation with discontinuous boundary conditions gives rise to specific problems of singular analysis, here, of edge problems in "simplest" form. Further singular problems are induced in other models of physics or applied sciences.

5. Boundary conditions as a source of singular operators

We now specify the observations from the preceding Section on how non-trivial interface configurations are induced by boundary conditions. First of all the boundary may have singularities anyway. Then, e.g., in a cube or some other piecewise smooth domain the non-smooth geometry on the boundary generates replacing the standard local coordinates by local models with the corresponding stratification, and then we need to activate the "technical" details of the singular analysis from the very beginning. However, if the boundary is smooth, the interfaces Z appearing, say, in jumping boundary conditions such as the Zaremba problem can be of a very "non-standard" behaviour. The interface Z in any case may correspond to various physical contents, e.g., in crack theory where jumping boundary conditions along Z correspond to singular stress-displacement factors of cracks in media see, e.g., [19] or examples in the monographs [10], [11], [16], [28], or other detailed descriptions in expositions devoted to applications for instance [10], [16], [19]. Simplest models of cracks can be described by one-dimensional intervals, when in local representation of the above-mentioned boundary, say, by $\mathbb{R}^{n-1}$, the crack is an interval $-1 \leq x_1 \leq 1$, with end points $x_1 = -1, x_1 = 1$, in the (x_1, x_2) hyperplane. Then we can interprete those end points as different conical singularities, and the open interval $-1 < x_1 < 1$ as an edge of codimension $n-2$. Every configurationof this kind, also when we admit other codimensions of the crack, or when we admit once again singularities along the crack, gives rise to a singular problem, i.e., the task of the mathematical approach is to create structures which are flexible enough to reflect the corresponding situation of the concrete model. This is what we mean by the influence of singular interfaces to the mathematical background.

Before we come to the structure of new inventions of the singular analysis which are considered in the following Sections, we note that other models of immediate practical interest, namely from many-particle systems in classical mechanics or quantum mechanics, see, e.g., papers [7] and [8] can also be seen from the point of view of singular analysis.

6. Singular spaces and degenerate operators

It has been a real challenge in the development of mathematical tools to describe the large variety of singular spaces by a transparent framework. The same is true of the variety of operators. Those are admitted to be of arbitrary order, since the calculus requires compositions and properties of solutions are identified with corresponding structures of parametrices, in the elliptic case. Then, in elaborating step by step the concept, we are aware of more and more specific structures which may be quite unexpected at first glance. Here in the present exposition we focus on interface aspects, i.e., singularities which are regarded as interfaces in situations coming from mixed or transmission problems but with singularities. Therefore, it seems to be the best way to once again indicate the formulation of singular spaces of the category $\mathfrak{M}_k$ for $k \geq 1$. A topological space M belongs to $\mathfrak{M}_k$, if M contains a "singular stratum" $s_k(M)$ such that

$$M \setminus s_k(M) \in \mathfrak{M}_{k-1} \tag{6.1}$$

has the structure of a (locally trivial) B^{Δ}_{k-1}-bundle over $s_k(M)$ for some $B_{k-1} \in \mathfrak{M}_{k-1}$ (assumed to be already defined) cf. also [3].

Analogously as the above-mentioned regularity of transition maps we have the notion of isomorphy of objects B_{k-1} in $\mathfrak{M}_{k-1}$. This gives rise to isomorphy between objects $\mathbb{R} \times B_{k-1}$, since the latter space again belongs to $\mathfrak{M}_{k-1}$, and then transitions

$$\mathbb{R}_+ \times B_{k-1} \to \mathbb{R}_+ \times B_k$$

are required to be restrictions of isomorphisms $\mathbb{R} \times B_{k-1} \to \mathbb{R} \times B_{k-1}$ to $\mathbb{R}_+ \times B_{k-1}$. This process is compatible with quotient spaces in the definition of cones $B_{k-1}^{\triangle} = (\overline{\mathbb{R}}_+ \times B_{k-1})/(\{0\} \times B_{k-1})$ which gives the bundle structure over $s_{k-1}(M)$ a precise meaning. Now (6.1) allows us to define an $s_{k-1}(M) := s_{k-1}(M \setminus s_k(M))$, and we may start the process all over again. This process is finite and we get an sequence of strata

$$s(M) := (s_0(M), s_1(M), \dots, s_k(M)) \tag{6.2}$$

of strictly decreasing dimension where all components belong to $\mathfrak{M}_0$. Examples are straight cones with singular base manifolds or wedges and, in particular we have a precise structure of singular interfaces, e.g., embedded in boundaries. Other examples can be easily derived, e.g., if an $M \in \mathfrak{M}_k$ is embedded in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then $\mathbb{R}^n \setminus M$ is singular in our category as well.

Another important point is to specify the nature of differential operators A and their principal symbolic hierarchies

$$\sigma(A) := (\sigma_0(A), \sigma_1(A), \dots, \sigma_k(A)) \tag{6.3}$$

associated with (6.2). The components of $\sigma(A)$ are associated with those of the stratification $s(M)$ of the respective singular space M. The simplest way of organizing operators of order $\mu \in \mathbb{N}$ belonging to the calculus over M is to look at edge-degenerate differential operators close to $s_k(M) \ni y$, locally $y = (y_1, \dots, y_{q_k}) \in \mathbb{R}^{q_k}$, which are of the form

$$A = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y)(-r\partial_r)^j (rD_y)^\alpha \tag{6.4}$$

for families $a_{j\alpha}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_k}, \mathrm{Diff}_{\mathrm{deg}}^{\mu-(j+|\alpha|)}(B_{k-1}))$ where degerate differential operators are indicated by subscript deg, which are assumed to be introduced in the inductive step for $B_{k-1} \in \mathfrak{M}_{k-1}$ before. We assume here $q_k > 0$. Then

$$\sigma_k(A)(y, \eta) = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(0, y)(-r\partial_r)^j (r\eta)^\alpha \tag{6.5}$$

is an operator-valued symbol for $(y, \eta) \in T^* s_k(M) \setminus 0$, acting between Kegel spaces $\mathcal{K}^{s,\gamma}(B_{k-1}^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B_{k-1}^\wedge)$ for weights $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$. It can be easily checked that the spaces $\mathcal{K}^{s,\gamma}(B_{k-1}^\wedge)$ admit the action of a group $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ of isomorphisms

$$(\kappa_\delta u)(r, x) := \delta^{(b_{k-1}+1)/2} u(\delta r, x), \delta \in \mathbb{R}_+,$$

where $b_{k-1} = \dim B_{k-1}$ and that

$$\sigma_k(A)(y, \delta\eta) = \delta^\mu \kappa_\delta^{-1} \sigma_k(A)(y, \eta) \kappa_\delta^{-1}.$$

This is an operator-valued substitute of homogeneity of functions in the standard sense. In the case $q_k = 0$ which corresponds to a corner singularity $s_k(M)$ we have also a symbolic structure attached to the corner, in this case called the principal conormal symbol of A which is a family of operators

$$\sigma_M(A)(v) : H^s(B_{k-1}) \to H^{s-\mu,\gamma'-\mu}(B_{k-1})$$

for weights $\gamma' = (\gamma_1, \dots, \gamma_{k-1})$ and $v \in \Gamma_{\frac{b_{k-1}}{2}-\gamma_k}$. Then applying the procedure in an iterative manner yields the full principal symbolic hierarchy where $\sigma_0(A)$ has the meaning of the standard homogeneous principal symbol of A over $s_0(M)$.

7. Edge- and corner quantizations

One of the supporting principles in analysis on singular space is to follow a step by step procedure from lower to higher orders of singularities. Spaces B with conical or edge singularities belong to $\mathfrak{M}_1$ while smooth manifolds X are elements of $\mathfrak{M}_0$. The quantizations start with the class

$$L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\zeta) \tag{7.1}$$

of classical pseudo-differential operators of order $\mu \in \mathbb{R}$, on compact $X \in \mathfrak{M}_0$ depending on parameters ζ, and their dimension depends on further iterations. The space (7.1) is Fréchet in a natural way, and we have the space $\mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\zeta))$ of holomorphic functions in $\mathbb{C}$ in the complex variable v with values in this space. This gives rise to the first essential quantization step, namely, we pass to the space

$$M^\mu_{\mathcal{O}_v}(X; \mathbb{R}^d_\zeta) \tag{7.2}$$

of all those holomorphic functions $h_1(v, \zeta) \in \mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\zeta))$ such that $h_1(\beta + i\rho, \zeta) \in L^\mu_{\mathrm{cl}}(X; \Gamma_\beta \times \mathbb{R}^d_\zeta)$ for every real β, uniformly in compact β-intervals.

Here

$$\Gamma_\beta := \{v \in \mathbb{C} : \mathrm{Re}\, v = \beta\}$$

is a weight line, motivated by the Mellin transform

$$Mu(v) = \int_0^\infty r^{v-1} u(r) dr,$$

first applied to compactly supported functions on $\mathbb{R}_+$, later also applied for vector- or operator- valued functions. We have in this context also weighted Mellin pseudo-differential operators, operating along the variable $r \in \mathbb{R}_+$, interpreted as a distance variable to the next higher singularities, e.g., when X is the base of the cone X^Δ, and we have in mind a $B \in \mathfrak{M}_1$ which is locally close to its edge to $Y = s_1(B)$ of dimension q_1 modelled on

$$X^\Delta \times \mathbb{R}^q \tag{7.3}$$

in the splitting of variables (r, x, y) which are variables in the open stretched wedge $X^\wedge \times \mathbb{R}^q$ for $X^\wedge = \mathbb{R}_+ \times X$. Weighted Mellin pseudo-differential operators with parameters have amplitude functions

$$h(r, y, v, \eta, \zeta) := \tilde{h}(r, y, v, r\eta, r\zeta) \tag{7.4}$$

for

$$\tilde{h}(r, y, v, \tilde{\eta}, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}_v}(X; \mathbb{R}^{q+d}_{\tilde{\eta},\tilde{\zeta}})) \tag{7.5}$$

and for $n = \dim X$ are of the form

$$\mathrm{Op}_{M_r}^{\gamma-n/2}(h)(y,\eta,\zeta)u = M_{v\to r}^{-1} h(r,y,v,\eta,\zeta) M_{r'\to v} u(r'), \tag{7.6}$$

where M stands for the weighted Mellin transform of weight $\gamma \in \mathbb{R}$, which acts as

$$u(r) \to (Mu)(v)|_{\Gamma_{\frac{n+1}{2}-\gamma}}.$$

Then the multiplication by f is combined with the point wise action of f when it is operator-valued, and the weighted inverse of the Mellin transform is defined as

$$g(v) \to \int_{\Gamma_{\frac{n+1}{2}-\gamma}} r^{-v} g(v) đv$$

for $đv = (2\pi i)^{-1}dv$ with integration on the weight line $\Gamma_{\frac{n+1}{2}-\gamma}$ from $\mathrm{Im}\, v = -\infty$ to $\mathrm{Im}\, v = +\infty$. Local parameter-dependent amplitude functions along Y in variables $y \in \mathbb{R}^q$ are of the form

$$a_1(y,\eta,\zeta) = h_1(y,\eta,\zeta) + (m_1 + g_1)(y,\eta,\zeta) \tag{7.7}$$

for

$$h_1(y,\eta,\zeta) = r^{-\mu}\{\omega_1 \mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta,\zeta)\omega_1'\},$$

where ω_1, ω_1' are cut-off functions in r and $m_1 + g_1$ represent the asymptotic content of the edge calculus. We will discuss this contribution later on.

8. The symbolic hierarchies associated with singular strata

Let $B \in \mathfrak{M}_1$ be a manifold with edge Y, near Y in local coordinates $y \in \mathbb{R}^q, q > 0$, modelled on

$$X^\Delta \times \mathbb{R}^q$$

for a closed $X \in \mathfrak{M}_0$. In stretched coordinates operators refer to the splitting of variables $(r,x,y) \in X^\wedge \times \mathbb{R}^q$. The space of edge operators

$$L^\mu(B, \boldsymbol{g}, \mathbb{R}^d_\zeta) \tag{8.1}$$

with parameters $\zeta \in \mathbb{R}^d$ and weight data $\boldsymbol{g} := (\gamma, \gamma-\mu, \Theta)$ and a weight interval $\Theta = (-(\theta+1), 0]$ is defined as the set of all

$$A(\zeta) = H(\zeta) + (M+G)(\zeta) + A_{\mathrm{int}}(\zeta) + G(\zeta). \tag{8.2}$$

The ingredients of (8.2) will be analyzed below. In any case after the complete definition we will have

$$L^\mu_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^d_\zeta) \subset L^\mu(B, \boldsymbol{g}; \mathbb{R}^d_\zeta)$$

and hence $A(\zeta)$ has a principal symbol

$$\sigma_0(A(\zeta))$$

of the standard parameter-dependent calculus of pseudo-differential operators over the smooth manifold $B \setminus Y$. However, close to Y where the part $H(\zeta) + (M+G)(\zeta)$ is localized, there is contributed a parameter-dependent operator valued principal edge symbol

$$\sigma_1(A(\cdot))(y,\eta,\zeta).$$

In order to give an idea on its nature we briefly describe symbol spaces of the classes

$$S^{\mu}(\mathbb{R}^q_y \times \mathbb{R}^{q+d}_{\eta,\zeta}; H, \tilde{H}) \text{ and } S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_y \times \mathbb{R}^{q+d}_{\eta,\zeta}; H, \tilde{H}) \tag{8.3}$$

which are generalizations of Hörmander's symbol spaces where subscript "cl" indicates classical symbols. In this notation H is a Hilbert space with group action, i.e., there is a family $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ of isomorphisms

$$\kappa_\delta : H \to H \tag{8.4}$$

such that $\kappa_\delta \kappa_{\delta'} = \kappa_{\delta\delta'}, \kappa_1 = \mathrm{id}_H$, and for every $h \in H$ the function $\delta \to \kappa_\delta h$ belongs to $C(\mathbb{R}_+, H)$. Concerning $\tilde{H}$ we make similar assumptions, relative to another $\tilde{\kappa} = \{\tilde{\kappa}_\delta\}_{\delta \in \mathbb{R}_+}$, cf. [26]. The notion also admits Fréchet spaces with group action, or finite-dimensional spaces $\mathbb{C}^N$ equipped with $\kappa_\delta = \mathrm{id}_{\mathbb{C}^N}$ for all $\delta \in \mathbb{R}_+$. The symbolic estimates of elements $a(y, \eta, \zeta)$ in the Hilbert space case have the form

$$\|\tilde{\kappa}_\delta^{-1}\{D_y^\alpha D_{\eta,\zeta}^\beta a(y, \eta, \zeta)\}\kappa_\delta\|_{\mathcal{L}(H,\tilde{H})} \le c\langle \eta, \zeta\rangle^{\mu - |\beta|}$$

for all multi-indices $\alpha \in \mathbb{N}^q, \beta \in \mathbb{N}^{q+d}, y \in K$ for compact $K \Subset \mathbb{R}^q$, with constants $c = c(\alpha, \beta, K)$. Here $\langle \eta, \zeta \rangle = (1 + |\eta|^2 + |\zeta|^2)^{1/2}$ and "classical" indicates twisted homogeneity, i.e., symbols have components

$$a_{(\mu-j)}(y, \eta, \zeta) \in S^{(\mu-j)}(\mathbb{R}^q \times (\mathbb{R}^{q+d} \setminus \{0\}); H, \tilde{H})$$

such that for every $N \in \mathbb{N}$

$$a(y, \eta, \zeta) - \sum_{j=0}^{N} \chi(\eta, \zeta) a_{(\mu-j)}(y, \eta, \zeta) \in S^{\mu-(N+1)}(\mathbb{R}^q \times (\mathbb{R}^{q+d} \setminus \{0\}), H, \tilde{H})$$

for any excision function $\chi(\eta, \zeta)$ in covariables and parameters in $\mathbb{R}^{q+d}$. Twisted homogeneity of order $\nu \in \mathbb{R}$ defines the space of all

$$f_{(\nu)}(y, \eta, \zeta) \in C^\infty(\mathbb{R}^q \times (\mathbb{R}^{q+d} \setminus \{0\}), \mathcal{L}(H, \tilde{H}))$$

such that

$$f_{(\nu)}(y, \delta\eta, \delta\zeta) = \delta^\nu \tilde{\kappa}_\delta^{-1} f_{(\nu)}(y, \eta, \zeta)\kappa_\delta$$

for all $(\eta, \zeta) \neq 0, \delta \in \mathbb{R}_+$. The operator-valued symbols $a(y, \eta, \zeta)$ in (8.3) open corresponding classes of pseudo-differential operators via Fourier quantization

$$\mathrm{Op}_y(a)(\zeta)u = \iint e^{i(y-y')\eta} a(y, \eta, \zeta) u(y') dy' đ\eta.$$

The argument functions belong to vector-valued weighted Sobolev spaces which are studied in more detail below in Section 9. The summands $A_{\mathrm{int}}(\zeta)$ in (8.2) belong to $L^\mu_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^d_\zeta)$ while $G(\zeta)$ are specific parameter-dependent smoothing operators. Those are described in [27].

After having completed explaining the edge operator spaces (8.1) we are in a similar position as before with (7.1) when we want to carry out the step to the next higher singularity. As essential ingredient was the space (7.2) on the level of $X \in \mathfrak{M}_0$ as base space. Now, (8.1) which refers to $B \in \mathfrak{M}_1$ allows us to introduce a space

$$M^\mu_{\mathcal{O}}(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta) \tag{8.5}$$

later on involved in holomorphic Mellin symbols for the next higher singular operators. The space (8.5) consists of all

$$h_2(v_2, \zeta) \in \mathcal{A}(\mathbb{C}_{v_2}, L^\mu(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta))$$

such that

$$h_2(\beta + i\rho_2, \zeta) \in L^\mu(B, \boldsymbol{g}_B; \Gamma_\beta \times \mathbb{R}^d_\zeta)$$

for every real β, uniformly in compact β-intervals. Note that in this process the dimension d can be determined in a new way and make it depending on B. The former $v \in \mathbb{C}$ from this moment on will be called v_1 and edge variables/covariables are changed to $y_1, \eta_1 \in \mathbb{R}^{q_1}$, with q_1 rather than q. Then the next higher edge calculus is based on

$$B^\Delta \times \mathbb{R}^{q_2}$$

and, similarly as before we look at the open stretched wedge $B^\wedge \times \mathbb{R}^{q_2}$ splitting of variables (r_2, x_2, y_2). It is clear now how we can organize analogues of Mellin symbols and Mellin operators (7.4), (7.5) and (7.6), respectively, and again we can form amplitude functions (7.7) close to $r_2 = 0$ by a similar process as before. Cut-off functions ω_2 and ω'_2 now refer to the new axial variables r_2 and r'_2, respectively. We do not repeat all elements of this approach, again, but it is demonstrated in which way we reach higher operator spaces over spaces in $\mathfrak{M}_2$, then $\mathfrak{M}_3$, etc.

9. Weighted cone- and wedge spaces

The operator spaces which we constructed so far are closely related to some weighted distribution spaces over infinite (stretched) cones $B^\wedge$ or wedges $B^\wedge \times \mathbb{R}^q$. Here, for keeping the ideas more transparent, we return to the simpler notation from the very beginning with $X \in \mathfrak{M}_0, B \in \mathfrak{M}_1$, and we refer to variables (r, x, y), etc. The wedge calculus on this level of consideration as well as all higher levels of singularity depend in a crucial way on some new inventions, which already have been introduced in the early days of edge calculus in [26], after investigation of [25]. Let us illustrate the way of creating weighted Sobolev spaces on manifolds $B \in \mathfrak{M}_1$ with edge. The local description of B close to an edge Y of dimension $q > 0$ is locally in variables based on the abstract concept in terms of Hilbert spaces H with group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$. Let $\mathcal{W}^s(\mathbb{R}^q, H), s \in \mathbb{R}$, denote the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} := \Big\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|^2_H đ\eta \Big\}^{1/2} \tag{9.1}$$

with $\hat{u}$ being the Fourier transform in $\mathbb{R}^q$, cf. [26], [27]. The spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ have a number of useful properties and admit similar functional-analytic considerations as the standard Sobolev spaces $H^s(\mathbb{R}^n)$ of smoothness s in $\mathbb{R}^n$. When we insert $H := H^s(\mathbb{R}^n)$ endowed with the group action

$$(\kappa_\delta u)(x) := \delta^{n/2} u(\delta x), \delta \in \mathbb{R}_+,$$

then a simple computation shows that

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^n)) = H^s(\mathbb{R}^{q+n}). \tag{9.2}$$

In the edge calculus we build up the local spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ by means of what we call weighted Kegel spaces $H = \mathcal{K}^{s,\gamma}(X^\wedge)$. Those are defined by

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{u = \omega u_0 + (1-\omega)u_\infty : u_0 \in \mathcal{H}^{s,\gamma}(X^\wedge), u_\infty \in H^s_{\mathrm{cone}}(X^\wedge)\}.$$

Here ω is any cut-off function on the r half-axis, $\equiv 1$ close to $r = 0$, $\equiv 0$ far from $r = 0$, and the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ are weighted cone Sobolev spaces, based on the Mellin transform in r and locally in x in variables $x \in \mathbb{R}^n$ on the Fourier transform. The norm in local terms is defined as

$$\|u\|_{\mathcal{H}^s(\mathbb{R}_+\times\mathbb{R}^n)} := \Big\{ \int_{\mathbb{R}^n} \int_{\Gamma_{\frac{n+1}{2}}} \langle v, \xi\rangle^{2s} |(F_{x\to\xi} M_{r\to v} u)(v,\xi)|^2 đv đ\xi \Big\}^{1/2}$$

for any $s, \gamma \in \mathbb{R}$. Then, globally along compact $X \in \mathfrak{M}_0$ we obtain the spaces by using an open covering of X by charts mapping to $\mathbb{R}^n$, a subordinate partition of unity and a corresponding summation in a standard way. The spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ are different from $\mathcal{H}^{s,\gamma}(X^\wedge)$ with respect to their behaviour for $r \to \infty$. The Kegel spaces to not feel the weight at ∞. The corresponding spaces $H^s_{\mathrm{cone}}(X^\wedge)$ treat the infinite stretched cone as a manifold with conical exit to ∞. A simple case which also can be used for the definition in the general case is $X := S^n$, the unit sphere in $\mathbb{R}^{1+n}$ with $(S^n)^\wedge$ being identified with $\mathbb{R}^{1+n} \setminus \{0\}$ via polar coordinates. In this case, after cutting out the origin we have

$$(1-\omega)H^s_{\mathrm{cone}}((S^n)^\wedge) = (1-\omega)H^s(\mathbb{R}^{1+n}).$$

This property has the consequence that, together with identity (9.2), the norms (9.1) for $H = \mathcal{K}^{s,\gamma}(X^\wedge)$ just produce spaces contained in $H^s_{\mathrm{loc}}(\mathbb{R}^{n+1+q})$. In fact, a careful evaluation of norms (9.1) shows that the local behaviour of norms for growing $|\eta|$ draws more and more $H^s(\mathbb{R}^{1+n})$-information to arbitrary small neighbourhoods $r < \epsilon$ for any $\epsilon > 0$, which induces the mentioned H^s_{comp}-identification for $r > \epsilon$ but which also shows that the more we approach $r = 0$ the norm expression feels the influence of the weight γ, cf. [27]. This makes the analysis with distributions in $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ a bit unusual. If B is a global manifold with edge we need a careful consideration on the transition behaviour under changing charts, connected with coordinate neighbourhoods close to the edge Y, cf., e.g., [27], but then, using a corresponding system of charts

$$\chi_j : \mathbb{R}^q \to Y, j = 0, \dots, N,$$

distributions in the global analogues of the norm of $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ supported close to Y, now denoted as elements in $H^{s,\gamma}(B)$, are defined by

$$\|u\|_{H^{s,\gamma}(B)} := \big\{ \sum_{j=1}^N \|(\varphi_j u) \circ (\mathrm{id}_{\mathbb{R}_+} \times (\chi_j^{-1})_*)\|^2_{\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))} \big\}^{1/2}$$

for a subordinate partition of unity $(\varphi_1, \dots, \varphi_N)$. By construction we then have

$$H^{s,\gamma}(B) \subset H^s_{\mathrm{loc}}(B \setminus Y).$$

The system of notation is that as soon we want to express local distributions, in the norms either in $\mathbb{R}^q$ or along $\mathbb{R}_+$ with the rescaling effect from the group action we employ calligraphic letters, but on (often compact) manifolds B with singularities we employ notation like $H^{s,\gamma}(B)$, analogously as standard Sobolev spaces, but γ only affects distributions in an arbitrary small neighbourhood of the singularity.

It is also essential to compare Mellin actions in $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)$-spaces with respect to $r \to \infty$, where a special choice of Mellin quantizations plays a role, together with the kernel cut-off method and the interplay with the pseudo-differential calculus on manifolds with conical exit to ∞.

10. Asymptotics contributed by elliptic problems

As noted before the strata of a space $M \in \mathfrak{M}_k$ may be interpreted as a generalization of interfaces of different dimension. In the zero-dimensional case we also speak about corners or conical singularities. A starting point of explaining the various levels of higher singular calculus may be differential operators in stretched coordinates, namely,

$$(r, x, y) \in B^\wedge \times \mathbb{R}^q \tag{10.1}$$

for a splitting of variables $(r, x) \in B^\wedge, B \in \mathfrak{M}_{k-1}$ and $y \in \mathbb{R}^q, q > 0$. In the corner case we have $q = 0$. Let us consider, for instance, the case $q > 0$. Then differential operators of order $\mu \in \mathbb{N}$ which reflect a higher geometric singularity, modeled on an edge with local variables $y \in \mathbb{R}^q$, are of the form (6.4). The general task is then to express by means of an adequate singular analogue of the smooth pseudo-differential calculus ellipticity, parametrices and regularity of solutions u of equations $Au = f$ for prescribed right-hand sides f. The problem is similar to the case of elliptic boundary value problems which are locally considered in variables

$$(r, y) \in \mathbb{R}_+ \times \mathbb{R}^q$$

where the wedge $\overline{\mathbb{R}}_+ \times \mathbb{R}^q$ is regarded as the local model of the respective manifold with boundary. In this case B is of dimension zero. As is well-known even this case is rich of phenomena which can make the program to a relatively voluminous calculus, see [2], [22] or [24] when the symbols $\sigma_0(A)$ have the transmission property at the boundary. In this case we can control solutions with Sobolev space regularity or, alternatively, with smoothness, up to the boundary. Except for the fact that we usually pose elliptic boundary conditions for vanishing or non-vanishing Atiyah-Bott abstruction, cf. [1], [30], we have to admit that generically the transmission property is violated, and then problems are to be embedded into the edge calculus. In this case it is more adequate to pass to weighted Sobolev spaces with weights $\gamma \in \mathbb{R}$ at $r = 0$. The choice of weights in the case of operators (6.4) depends on the individual operator A, mainly on the principal conormal symbol subordinate to (6.5)

$$\sigma_M(A)(y, v) := \sum_{j=0}^{\mu} a_{j0}(0, y) v^j : H^s(X) \to H^{s-\mu}(X) \tag{10.2}$$

which is in this case a polynomial in $v \in \mathbb{C}$. Recall that the relation between v and the Fuchs type derivative $-r\frac{\partial}{\partial r}$ comes from the Mellin transform. For pseudo-differential A we have $\sigma_M(A)(y, \eta) = \tilde{h}(0, y, v, 0) + f_0(y, v)$ where the first summand is defined by (10.4) below and $f_0(y, v)$ is a so-called smoothing Mellin symbol. In solving elliptic equations we also have to invert (10.2) and this can only work on weight lines $\Gamma_{\frac{n+1}{2}-\gamma} \ni v$ where (10.2) does not vanish. Thus, since (10.2) is a polynomial in v, there are only finitely many exceptional weights such that this condition is violated. In the pseudo-differential case there may be infinitely many. Assuming that those are independent of y there is the question on their precise position. In weighted Mellin Sobolev spaces $\mathcal{H}^{s,\gamma}$ we usually can observe more specific properties, namely, subspaces with asymptotics. In order to illustrate the ideas for establishing asymptotic information of solutions of $Au = f$ in the higher case (6.4) we assume for simplicity $k = 1$, i.e., the standard edge-degenerate case. Then the operator can be close to the edge written in coordinates $y \in \mathbb{R}^q$ in the form

$$A = \mathrm{Op}_y\{r^{-\mu}\omega \mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\omega'\} \tag{10.3}$$

where analogously as (7.6), here for the moment without parameter ζ, the Mellin symbol h has the form

$$h(r,y,v,\eta) = \tilde{h}(r,y,v,r\eta)$$

for an

$$\tilde{h}(r,y,v,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M_{\mathcal{O}}^{\mu}(X;\mathbb{R}^q_{\tilde{\eta}})). \tag{10.4}$$

Then solutions are charecterized via a parametrix construction which starts with a test parametrix

$$P_1 := \mathrm{Op}_y\{\omega \mathrm{Op}_M^{\gamma-\mu-n/2}(h^{(-1)})(y,\eta)r^{\mu}\omega'\}$$

for a Mellin symbol

$$h^{(-1)}(r,y,v,\eta) = \tilde{h}^{(-1)}(r,y,v,r\eta)$$

for an

$$\tilde{h}^{(-1)}(r,y,v,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M_{\mathcal{O}}^{-\mu}(X;\mathbb{R}^q_{\tilde{\eta}})). \tag{10.5}$$

We assume here ellipticity which contains the condition that $\tilde{h}(r,y,v,\tilde{\eta})$ takes values in parameter-dependent elliptic elements of $M_{\mathcal{O}}^{\mu}(X;\mathbb{R}^q_{\tilde{\eta}})$. Then we find $h^{(-1)}(r,y,v,\eta)$ in such a way that

$$P_1 A = 1 + M + G \tag{10.6}$$

for a Mellin operator M with smoothing Mellin symbols who contain asymptotic information and a Green operator in the edge calculus. Now we modify P_1 by multiplying by a factor from the left of the form $(1 + L)$ for another smoothing Mellin operator, i.e., pass to

$$P_2 = (1 + L)P_1. \tag{10.7}$$

The construction of L employs relations on the level of principal conormal symbols and employs the identity

$$\sigma_M(P_1)(y,v)\sigma_M(A)(y,v) = 1 + \sigma_M(M)(y,v).$$

There is another smoothing Mellin operator L such that its leading conormal symbol $\sigma_M(L)$ satisfies relation

$$(1 + \sigma_M(L)(y,v))(1 + \sigma_M(M)(y,v)) = 1.$$

This implies

$$(1+\sigma_M(L)(y,v))\sigma_M(P_1)(y,v)\sigma_M(A)(y,v)=1$$

and hence

$$(\sigma_M(A)(y,v))^{-1}=(1+\sigma_M(L)(y,\eta))\sigma_M(P_1)(y,v).$$

Then, on the level of operators we form P_2 for

$$\sigma_M(P_2)(y,v):=(1+\sigma_M(L)(y,v))\sigma_M(P_1)(y,v)$$

which gives us a parametrix with a refined property (10.6), namely, we have

$$P_2A=1+N+G$$

for a smoothing Mellin operator N and another Green operator G such that

$$\sigma_M(N+G)=0.$$

Then a formal Neumann series argument gives us a P_3 such that

$$P_3A=1+G$$

for a smoothing, so-called Green operator. Similar technique for constructing parametrices in the parabolic case has been applied in [18]. So far we did not indicate in this exposition the nature of Green and smoothing Mellin operators which encode asymptotics. In any case in the operator algebra on a manifold $M\in\mathfrak{M}_1$ with edge $s_1(M)=Y$, locally close to Y modelled on $X^\Delta\times\mathbb{R}^q$ for some compact closed $X\in\mathfrak{M}_0$, operators are locally in $y\in\mathbb{R}^q$ described modulo smoothing operators and interior operators in $L^\mu_{\mathrm{cl}}(s_0(M))$ by operator-valued amplitude functions, namely,

$$a(y,\eta)=r^{-\mu}\omega\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\omega'+(m+g)(y,\eta), \tag{10.8}$$

cf. (10.3) and Mellin and Green symbols $m(y,\eta)+g(y,\eta)$. The smoothing Mellin symbols are finite linear combinations of expressions like

$$r^{-\mu}\omega_\eta r^j\mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)\omega'_\eta \tag{10.9}$$

from $j=0,\dots,\theta$ for a weight interval $\Theta=(-(\theta+1),0],\theta\in\mathbb{N}$, with Mellin symbols $f_{j\alpha}\in M^{-\infty}_{\mathcal{R}_{j\alpha}}(X)$. If $M\in\mathfrak{M}_1$ is our manifold with edge of dimension $q>0$, we have

$$L^\mu(M,\boldsymbol{g}), \tag{10.10}$$

for weight data $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$, the space of all operators of the form

$$A=H+M+G+A_{\mathrm{int}}+C$$

where H is a non-smoothing Mellin operator with amplitude function

$$a_h(y,\eta)=r^{-\mu}\omega\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\omega',$$

moreover, M is a smoothing Mellin operator, defined amplitude functions by (10.9), and G is a Green operator. All those contributions are localized close to Y. Moreover, we assume $A_{\mathrm{int}}\in L^\mu_{\mathrm{cl}}(M\setminus Y)$ and C is a global smoothing operator. By asymptotics of solutions $u(r,y)$ to elliptic equations $Au=f$ in simplest cases we understand that u admits a representation

$$u(r,y)=u_{\mathrm{sing}}(r,y)+u_{\mathrm{flat}}(r,y)$$

where u_{sing} is a singular function

$$u_{\text{sing}} \in \mathcal{E}_{\mathcal{P}}(X^\wedge)$$

with $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ being the space of functions

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) = \{\omega \sum_{j=1}^{N} \sum_{k=0}^{m_j} c_{jk} r^{-p_j} \log^k r : c_{jk} \in C^\infty(X)\} \tag{10.11}$$

for a cut-off function $\omega(r)$ and $p_j \in \mathbb{C}$, which form together with $m_j \in \mathbb{N}$ a sequence

$$\mathcal{P} = \{(p_j, m_j)\}_{j\in\mathbb{J}} \subset \mathbb{C} \times \mathbb{N},$$

called a discrete asymptotic type, where $\pi_{\mathbb{C}}\mathcal{P} = \{p_j\}_{j\in\mathbb{J}}$ is finite when the weight interval $\Theta = ((-\theta + 1), 0]$ is finite, and otherwise for $\Theta = (-\infty, 0]$ and infinite $\pi_{\mathbb{C}}\mathcal{P}$ we ask $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$. In any case it is assumed that

$$\pi_{\mathbb{C}}\mathcal{P} \subset \{z : \frac{n+1}{2} - \gamma - (\theta + 1) < \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$$

for the reference weight $\gamma \in \mathbb{R}$. Flat functions are elements of

$$\mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge) = \varprojlim_{\epsilon>0} \mathcal{K}^{s,\gamma-(\theta+1)-\epsilon}(X^\wedge).$$

If singular functions belong to (10.11) we assume the points p_j to be independent of y. Otherwise we have variable discrete asymptotics. This can be described in terms of continuous asymptotics which is for finite Θ encoded by singular functions of the form

$$u(r, y) = \omega\langle\zeta, r^{-z}\rangle$$

for a function ζ with values in analytic functionals

$$\zeta(y) \in C^\infty(\mathbb{R}^q, \mathcal{A}'(K, C^\infty(X))) \tag{10.12}$$

for a compact set $K \subset \{z \in \mathbb{C} : \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$. Properties of analytic functionals in connection with asymptotics can be found in [16]. Basics are also elaborated in [28]. For instance, if $f(y, z)$ is a family of, say, $C^\infty(X)$-valued meromorphic functions where the poles belong to the compact set K and which is smooth in $y \in \mathbb{R}^q$ as a function with values in $\mathcal{A}(\mathbb{C} \setminus K, C^\infty(X))$, then if C is a counter clockwise smooth curve surrounding K, the expression

$$\langle\zeta_f(y), h\rangle := \int_C f(y, z)h(z)đz,$$

$h \in \mathcal{A}(\mathbb{C})$, defines a function (10.12) which is pointwise discrete and of finite order. The poles may depend on y and also change multiplicities. This is a typical phenomenon in solutions when the non-bijectivity points of (10.2) on the left of the reference weight line $\Gamma_{\frac{n+1}{2}-\gamma}$ are not constant with respect to y. In the calculus the above-mentioned Mellin asymptotic types $\mathcal{R}_{j\alpha}$ also may depend on y in a similar manner represented by suitable closed sets in the complex Mellin plane. More material can also be found in [14] or [27].

Let us return to our elliptic differential operator (10.3), and assume ellipticity in the sense that the interior symbol is elliptic over $M \setminus Y$ and the reduced interior symbol is also elliptic and the symbol (10.4) admits a parametrix (10.5) and that in addition (10.2) is bijective on the weight line $\Gamma_{\frac{n+1}{2}-\gamma}$ and also that

$$\sigma_1(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{10.13}$$

takes values in isomorphisms for all $y \in X$ and $\eta \neq 0$ (this condition is independent of s). Then we have

Theorem 10.1. *An elliptic A defines a Fredholm operator*

$$A : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M)$$

for every $s \in \mathbb{R}$ and has a parametrix $P \in L^{-\mu}(M, \boldsymbol{g}^{-1})$ for $\boldsymbol{g}^{-1} = (\gamma-\mu, \gamma, \Theta)$. For every asymptotic type $\mathcal{Q}$ there is an asymptotic type $\mathcal{P}$ such that $Au \in H^{s-\mu,\gamma-\mu}_{\mathcal{Q}}(M)$ implies $u \in H^{s,\gamma}_{\mathcal{P}}(M)$.

This result has several modifications, in particular when we ask a weaker condition than (10.13). If, for instance (10.13) is only a family of Fredholm operators, then we can pose additional edge conditions which are analogues of Shapiro-Lopatinski elliptic conditions or Toeplitz conditions, see also [17] and [31].

Theorem (10.1) belongs to the typical qualitative results of the pseudo-differential analysis on a manifold with edge. In analogous form it is also meaningful on manifolds with conical singularities and also on infinite (stretched) cones $X^\wedge$. The same is true of spaces in $\mathfrak{M}_k$ for higher k. Results of this kind illustrate what we understand by the asymptotic content of analysis on spaces with singularities. For any $M \in \mathfrak{M}_k$ the subset N of singularities, mentioned in Section 2, just consists of $N = M \setminus s_0(M)$. We have $N \in \mathfrak{M}_{k-1}$, and hence N can also be the host of a corresponding calculus of less singularity order, and similarly $N \setminus s_0(N) \in \mathfrak{M}_{k-2}$, etc. On the other hand when we see M in relation with N, then the asymptotic information contained in solutions to elliptic equations or in parametrices is concentrated in any neighbourhood of N. Thus the singular subset N is "loaded" with asymptotic quantities, distributed on the strata of N of different levels of singularity. The investigation of this phenomenon is by no means finished. In fact it suggests further analysis on how asymptotic data are contained in the calculus which produces qualitative properties of solutions.

11. Concluding remarks

The analysis on manifolds with singularities has been created by the desire to do similar things for solvability of elliptic partial differential equations in the non-smooth case as on smooth manifolds. Many achievements of the classical analysis, e.g., of boundary value problems contributed to a new approach on singular manifolds. The recent development became a branch of analysis with operator-valued symbols, holomorphic and meromorphic Mellin symbols, and asymptotic phenomena. The present state of the development is that for every level $k \in \mathbb{N}$ of singularity we have an algebra of operators, analogously as (10.10), which contains interesting operators from applications and which itself is the starting point of further steps in a hierarchy of structures. In particular, the algebras can be generalized to algebras of parameter-dependent operators, with extra covariables $\zeta \in \mathbb{R}^d$ which can be activated in a next step for the symbolic background of operators on the next level of manifolds of singularities of degree $k+1$. Let us finally note that many questions remain for research in future, e.g., index theory in these operator algebras, or the investigation of additional conditions along the strata, e.g., of trace and potential type.

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