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# $k$-CF FUNCTIONS AND $\Box_b$ ON THE QUATERNIONIC HEISENBERG GROUP

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Abstract. The tangential $k$-Cauchy-Fueter operator and $k$-CF functions on the quaternionic Heisenberg group are quaternionic counterparts of the tangential CR operator $\bar{\partial}_b$ and CR functions on the Heisenberg group in the theory of several complex variables. We analyze the operator $\Box_b$ associated the tangential 2-Cauchy-Fueter operator and give its fundamental solution when the coefficients of the group satisfy the condition $\sum_{l=0}^{n-1} a_l \neq \pm \sum_{l=0}^{n-1} |a_l|$. As an application, we prove that the $L^p$-integrable 2-CF function space is trivial in this case. We also discuss the results for general $k$.

## 1. Introduction

In this paper we consider the $(4n+3)$-dimensional *quaternionic Heisenberg group* $\mathscr{H} = \mathbb{H}^n \oplus \mathrm{Im}\mathbb{H}$ equipped with the multiplication given by

$$(y,s)\cdot(y',s') = \left(y+y', s+s'+2\sum_{j=1}^{n} a_{j-1}\mathrm{Im}(\bar{y}_j y'_j)\right), \tag{1.1}$$

where $y, y' \in \mathbb{H}^{4n}$ and $s, s' \in \mathrm{Im}\mathbb{H}$. We are interested in the *tangential $k$-Cauchy-Fueter operator*

$$\mathscr{D}_b^{(k)} : C^\infty(\mathscr{H}, \odot^k\mathbb{C}^2) \to C^\infty(\mathscr{H}, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}) \tag{1.2}$$

$k = 1, 2, \cdots$, where $\odot^p\mathbb{C}^2$ is the $p$-th symmetric power of $\mathbb{C}^2$. They are tangential version of the $k$-Cauchy-Fueter operator, and the quaternionic counterparts of the tangential Cauchy-Riemann operator $\bar{\partial}_b$ over the Heisenberg group in the several complex variables. Corresponding to the notion of a CR function, the distribution $f$ satisfying $\mathscr{D}_b^{(k)} f = 0$ is called a *$k$-CF function*. Such functions for $k = 1$ are called *anti-CRF function* and play an important role in study of the psudo-Einstein equation over the quaternionic Heisenberg group [4] [5]. When $n = 1$ and $a_0 = 1$ (or any fixed number), as the CR case, we proved that the space of $L^2$-integrable $k$-CF functions $\mathcal{A}(\mathscr{H}, \mathbb{C}^{k+1}) = \{f \in L^2(\mathscr{H}, \mathbb{C}^{k+1}); \mathscr{D}_b^{(k)} f = 0\}$ on the quaternionic Heisenberg group is of infinite dimensional [7]. Here we identify $\odot^k\mathbb{C}^2$ with $\mathbb{C}^{k+1}$.

In [7], we use the group Fourier transform on the quaternionic Heisenberg group to analyze the operator

$$\Box_b := \mathscr{D}_b^* \mathscr{D}_b$$

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associated the tangential $k$-Cauchy-Fueter operator. We drop superscripts $(k)$ for simplicity. We also find the Szegö kernel of the *Szegö projection operator* $P : L^2(\mathscr{H}, \mathbb{C}^{k+1}) \to \mathcal{A}(\mathscr{H}, \mathbb{C}^{k+1})$. For higher dimensional quaternionic Heisenberg group, completely different phenomenon happens when

$$\sum_{l=0}^{n-1} a_l \neq \pm \sum_{l=0}^{n-1} |a_l|. \tag{1.3}$$

Namely the space of all $L^p$-integrable 2-CF functions

$$\mathcal{A}_p(\mathscr{H}, \mathbb{C}^3) = \{f \in L^p(\mathscr{H}, \mathbb{C}^3); \mathscr{D}_b^{(2)} f = 0\}, \quad 1 < p < \infty, \tag{1.4}$$

is trivial.

**Theorem 1.1.** $\mathcal{A}_p(\mathscr{H}, \mathbb{C}^3) = \{0\}$.

For $k = 2$, define

$$\ker \Box_b := \left\{ f \in L^p(\mathscr{H}, \mathbb{C}^3); \mathscr{D}_b f \in L^p(\mathscr{H}, \mathbb{C}^3), \mathscr{D}_b^* \mathscr{D}_b f = 0 \right\}. \tag{1.5}$$

where $\mathscr{D}_b^* \mathscr{D}_b f = 0$ in the sense of distributions. Obviously $\mathcal{A}_p(\mathscr{H}, \mathbb{C}^3) \subset \ker \Box_b$. We will give the fundamental solution of $\Box_b$ under the assumption (1.3).

The main idea to construct the fundamental solution of $\Box_b$ is as follows. We first calculate the explicit matrix form of $\Box_b$ and give its partial Fourier transformation $\widetilde{\Box}_{b;\tau}$. Then we diagonalize $\widetilde{\Box}_{b;\tau}$ for fixed $\tau \in \mathbb{R}^3 \setminus \{0\}$ so that the diagonal entries essentially have the form

$$H_\alpha = \alpha - \sum_{j=1}^{4n} \partial_{x_j}^2 + \sum_{j=1}^{4n} \lambda_j^2 x_j^2.$$

The fundamental solution of this operator was already given by Calin, Chang and Tie [1]. So we can use it to give the fundamental solution of $\widetilde{\Box}_{b;\tau}$ on $\mathbb{R}^{4n}$. At last we get the fundamental solution $K(y, s)$ of $\Box_b$ by using the inverse Fourier transformation.

$K(y, s)$ is a homogeneous distribution of order $-4n - 4$ on $\mathscr{H}$. $f \mapsto K * f$ is a bounded operator from $L^p$ to $L^q$ with $\frac{1}{q} = \frac{1}{p} - \frac{2}{Q}$, where $Q = 4n + 6$ is the homogeneous dimension of $\mathscr{H}$. Theorem 1.1 is a consequence of this result. We also give the result for $k = 3$. But we can not give result for general $k$, because we are not able to write down the matrix to diagonalize $\widetilde{\Box}_{b;\tau}$ for general $k$ (cf. Remark 3.1).

## 2. The tangential 2-Cauchy-Fueter operator and $\Box_b$

2.1. **The quaternionic Heisenberg group.** Recall that for $y = y_1 + y_2\mathbf{i}_1 + y_3\mathbf{i}_2 + y_4\mathbf{i}_3$ and $y' = y'_1 + y'_2\mathbf{i}_1 + y'_3\mathbf{i}_2 + y'_4\mathbf{i}_3$, we have

$$\begin{aligned} \operatorname{Im}(\bar{y}y') &= \operatorname{Im}\{(y_1 - y_2\mathbf{i}_1 - y_3\mathbf{i}_2 - y_4\mathbf{i}_3)(y'_1 + y'_2\mathbf{i}_1 + y'_3\mathbf{i}_2 + y'_4\mathbf{i}_3)\} \\ &= (y_1y'_2 - y_2y'_1 - y_3y'_4 + y_4y'_3)\mathbf{i}_1 + (y_1y'_3 - y_3y'_1 + y_2y'_4 - y_4y'_2)\mathbf{i}_2 \\ &\quad + (y_1y'_4 - y_4y'_1 - y_2y'_3 + y_3y'_2)\mathbf{i}_3 =: \sum_{\beta=1}^{3} \sum_{j,k=1}^{4} B^\beta_{kj} y_k y'_j \mathbf{i}_\beta \end{aligned} \tag{2.1}$$

(cf. (1.5) in [10]), where $B^{\beta}_{kj}$ is the $(k,j)$-th entry of the following matrices

$$B^1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B^2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$
$$B^3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

It is easy to see that matrices $B^1, B^2$ and $B^3$ satisfy the commutating relation of quaternions:

$$(B^1)^2 = (B^2)^2 = (B^3)^2 = -id, \qquad B^1B^2 = B^3. \tag{2.3}$$

By (2.1), the multiplication of the quaternionic Heisenberg group can written in terms of real variables as (cf. [9])

$$(y,s)\cdot(y',s') = \left( y + y', s_\beta + s'_\beta + 2\sum_{l=0}^{n-1}\sum_{j=1}^{4} a_l B^{\beta}_{kj} y_{4l+k} y'_{4l+j} \right),$$

where $\beta = 1,2,3,\ y = (y_1, \cdots, y_{4n}),\ y' = (y'_1, \cdots, y'_{4n}) \in \mathbb{R}^{4n},\ s = (s_1, s_2, s_3),\ s' = (s'_1, s'_2, s'_3) \in \mathbb{R}^3$. Then

$$Y_{4l+j} := \frac{\partial}{\partial y_{4l+j}} + 2a_l \sum_{\beta=1}^{3}\sum_{k=1}^{4} B^{\beta}_{kj} y_{4l+k} \frac{\partial}{\partial s_\beta}, \quad l = 0, 1, \cdots n-1,\ j = 1, \cdots, 4, \tag{2.4}$$

are standard left invariant vector fields on the quaternionic Heisenberg group $\mathscr{H}$, whose brackets are

$$[Y_{4l+k}, Y_{4l+j}] = 4a_l \sum_{\beta=1}^{3} B^{\beta}_{kj} \partial_{s_\beta}, \quad \text{and} \quad [Y_{4l+k}, Y_{4l'+j}] = 0, \quad l \neq l', \tag{2.5}$$

for $l, l' = 0, \cdots, n-1,\ j,k = 1, \cdots, 4$.

2.2. **The tangential 2-Cauchy-Fueter operator.** We consider complex left invariant vector fields

$$(Z_{AA'}) := \begin{pmatrix} Y_1 + iY_2 & -Y_3 - iY_4 \\ Y_3 - iY_4 & Y_1 - iY_2 \\ \vdots & \vdots \\ Y_{4l+1} + iY_{4l+2} & -Y_{4l+3} - iY_{4l+4} \\ Y_{4l+3} - iY_{4l+4} & Y_{4l+1} - iY_{4l+2} \\ \vdots & \vdots \end{pmatrix}$$

on $\mathscr{H}$, which are motivated by the embedding of quaternionic algebra $\mathbb{H}$ into $\mathbb{C}^{2\times 2}$ :

$$x_1 + x_2\mathbf{i}_1 + x_3\mathbf{i}_2 + x_4\mathbf{i}_3 \mapsto \begin{pmatrix} x_1 + ix_2 & -x_3 - ix_4 \\ x_3 - ix_4 & x_1 - ix_2 \end{pmatrix}.$$

The matrices

$$(\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\varepsilon^{A'B'}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are used to raise or lower primed indices, e.g. $Z_A^{A'} = \sum_{B'=0',1'} Z_{AB'}\varepsilon^{B'A'}$. Here $(\varepsilon^{A'B'})$ is the inverse of $(\varepsilon_{A'B'})$. Then $Z_A^{0'} = Z_{A1'}, Z_A^{1'} = -Z_{A0'}$, i.e.

$$\left(Z_A^{A'}\right) = \begin{pmatrix} -Y_3 - iY_4 & -Y_1 - iY_2 \\ Y_1 - iY_2 & -Y_3 + iY_4 \\ \vdots & \vdots \\ -Y_{4l+3} - iY_{4l+4} & -Y_{4l+1} - iY_{4l+2} \\ Y_{4l+1} - iY_{4l+2} & -Y_{4l+3} + iY_{4l+4} \\ \vdots & \vdots \end{pmatrix}$$

where $A = 0, 1, \cdots, 2n-1$, $A' = 0', 1'$. An element of $\mathbb{C}^2$ is denoted by $(f_{A'})$ with $A' = 0', 1'$. The symmetric power $\odot^2\mathbb{C}^2$ is the subspace of $\otimes^2\mathbb{C}^2$ whose element is a $2^2$-tuple $(f_{A'B'})$ with $A', B' = 0', 1'$, such that $f_{A'B'}$ are invariant under permutations of subscripts, i.e.

$$f_{A'B'} = f_{B'A'}.$$

The tangential 2-Cauchy-Fueter operator $\mathscr{D}_b$ in (1.2) is given by

$$(\mathscr{D}_b f)_{A'A} := \sum_{B'=0',1'} Z_A^{B'} f_{B'A'}, \tag{2.6}$$

(cf. (1.19) in [8]). We have isomorphisms

$$\odot^2\mathbb{C}^2 \cong \mathbb{C}^3, \qquad \mathbb{C}^2 \otimes \mathbb{C}^{2n} \cong \mathbb{C}^{4n}, \tag{2.7}$$

by identifying $f \in \odot^2\mathbb{C}^2$ and $F \in \mathbb{C}^2 \otimes \mathbb{C}^{2n}$ with

$$f := \begin{pmatrix} f_{0'0'} \\ f_{1'0'} \\ f_{1'1'} \end{pmatrix}, \quad F := \begin{pmatrix} F_{0'0} \\ \vdots \\ F_{0'(2n-1)} \\ F_{1'0} \\ \vdots \\ F_{1'(2n-1)} \end{pmatrix} \tag{2.8}$$

respectively. Then the tangential 2-Cauchy-Fueter operator is a $(4n \times 3)$-matrix valued differential operator:

$$\mathscr{D}_b = \begin{pmatrix} -Y_3 - iY_4 & -Y_1 - iY_2 & 0 \\ Y_1 - iY_2 & -Y_3 + iY_4 & 0 \\ \vdots & \vdots & \vdots \\ -Y_{4n-1} - iY_{4n} & -Y_{4n-3} - iY_{4n-2} & 0 \\ Y_{4n-3} - iY_{4n-2} & -Y_{4n-1} + iY_{4n} & 0 \\ 0 & -Y_3 - iY_4 & -Y_1 - iY_2 \\ 0 & Y_1 - iY_2 & -Y_3 + iY_4 \\ \vdots & \vdots & \vdots \\ 0 & -Y_{4n-1} - iY_{4n} & -Y_{4n-3} - iY_{4n-2} \\ 0 & Y_{4n-3} - iY_{4n-2} & -Y_{4n-1} + iY_{4n} \end{pmatrix} \tag{2.9}$$

(see (3.2) in [11] for similar operator on quaternionic space $\mathbb{H}^n$). But here our operator is defined over $\mathscr{H}$.

2.3. **The operator $\Box_b$.** By definition, the formal adjoint operator $Y_j^*$ of $Y_j$ is $-Y_j$, for $j = 1, \cdots, 4$, i.e.

$$\int_{\mathscr{H}} Y_j u \cdot \bar{v} \mathrm{d}y \mathrm{d}s = - \int_{\mathscr{H}} u \cdot \overline{Y_j v} \mathrm{d}y \mathrm{d}s,$$

for any $u, v \in C_0^\infty(\mathscr{H}, \mathbb{C})$. The formal adjoint operator of $\mathscr{D}_b$ is defined by

$$\int_{\mathscr{H}} \langle \mathscr{D}_b f, g \rangle_{\mathbb{C}^{4n}} \mathrm{d}y \mathrm{d}s = \int_{\mathscr{H}} \langle f, \mathscr{D}_b^* g \rangle_{\mathbb{C}^3} \mathrm{d}y \mathrm{d}s,$$

for any $f \in C_0^\infty(\mathscr{H}, \mathbb{C}^3), g \in C_0^\infty(\mathscr{H}, \mathbb{C}^{4n})$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}^p}$ is the Hermitian inner product of $\mathbb{C}^p$. Then $\mathscr{D}_b^* = -\overline{\mathscr{D}_b}^t$ is the $(3 \times 4n)$-matrix

$$= \left( \begin{matrix} Y_3 - iY_4 & -Y_1 - iY_2 \cdots & Y_{4n-1} - iY_{4n} & -Y_{4n-3} - iY_{4n-2} \\ Y_1 - iY_2 & Y_3 + iY_4 \cdots & Y_{4n-3} - iY_{4n-2} & Y_{4n-1} + iY_{4n} \\ 0 & 0 \cdots & 0 & 0 \end{matrix} \right. \tag{2.10}$$

$$\left. \begin{matrix} 0 & 0 \cdots & 0 & 0 \\ Y_3 - iY_4 & -Y_1 - iY_2 \cdots & Y_{4n-1} - iY_{4n} & -Y_{4n-3} - iY_{4n-2} \\ Y_1 - iY_2 & Y_3 + iY_4 \cdots & Y_{4n-3} - iY_{4n-2} & Y_{4n-1} + iY_{4n} \end{matrix} \right).$$

**Proposition 2.1.**

$$\Box_b = \Delta_b \cdot \begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} + 8a \begin{pmatrix} i\partial_{s_1} & \partial_{s_2} - i\partial_{s_3} & 0 \\ -\partial_{s_2} - i\partial_{s_3} & 0 & \partial_{s_2} - i\partial_{s_3} \\ 0 & -\partial_{s_2} - i\partial_{s_3} & -i\partial_{s_1} \end{pmatrix}, \tag{2.11}$$

*where* $\Delta_b := -\sum_{l=0}^{n-1} \sum_{j=1}^4 Y_{4l+j}^2$ *is the SubLaplacian and* $a = \sum_{l=0}^{n-1} a_l$.

*Proof.* It follows from (2.5) and matrix $B^1, B^2, B^3$ in (2.2) that

$$\begin{aligned} &[Y_{4l+1}, Y_{4l+2}] = -[Y_{4l+3}, Y_{4l+4}] = 4a_l \partial_{s_1}, \qquad [Y_{4l+1}, Y_{4l+3}] = [Y_{4l+2}, Y_{4l+4}] = 4a_l \partial_{s_2}, \\ &[Y_{4l+1}, Y_{4l+4}] = -[Y_{4l+2}, Y_{4l+3}] = 4a_l \partial_{s_3}, \end{aligned} \tag{2.12}$$

for $l = 0, 1, \cdots, n-1$. The product of $\mathscr{D}_b^*$ in (2.10) and $\mathscr{D}_b$ in (2.9) is given by

$$\begin{aligned}
(\Box_b)_{11} =& \sum_{l=0}^{n-1} \{(Y_{4l+3} - iY_{4l+4})(-Y_{4l+3} - iY_{4l+4}) \\
& \qquad\qquad\qquad + (-Y_{4l+1} - iY_{4l+2})(Y_{4l+1} - iY_{4l+2})\} \\
=& - \sum_{l=0}^{n-1} \sum_{j=1}^{4} Y_{4l+j}^2 - \sum_{l=0}^{n-1} \{i[Y_{4l+3}, Y_{4l+4}] - i[Y_{4l+1}, Y_{4l+2}]\} \\
=& \Delta_b + 8ai\partial_{s_1}, \\
(\Box_b)_{12} =& (\Box_b)_{23} = \sum_{l=0}^{n-1} \{(Y_{4l+3} - iY_{4l+4})(-Y_{4l+1} - iY_{4l+2}) \\
& \qquad\qquad\qquad + (-Y_{4l+1} - iY_{4l+2})(-Y_{4l+3} + iY_{4l+4})\} \\
=& \sum_{l=0}^{n-1} \{[Y_{4l+1}, Y_{4l+3}] + [Y_{4l+2}, Y_{4l+4}] - i[Y_{4l+1}, Y_{4l+4}] + i[Y_{4l+2}, Y_{4l+3}]\} \\
=& \, 8a\partial_{s_2} - 8ai\partial_{s_3}, \\
(\Box_b)_{21} =& (\Box_b)_{32} = \sum_{l=0}^{n-1} \{(Y_{4l+1} - iY_{4l+2})(-Y_{4l+3} - iY_{4l+4}) \\
& \qquad\qquad\qquad + (Y_{4l+3} + iY_{4l+4})(Y_{4l+1} - iY_{4l+2})\} \\
=& \sum_{l=0}^{n-1} \{-[Y_{4l+1}, Y_{4l+3}] - [Y_{4l+2}, Y_{4l+4}] - i[Y_{4l+1}, Y_{4l+4}] + i[Y_{4l+2}, Y_{4l+3}]\} \\
=& - 8a\partial_{s_2} - 8ai\partial_{s_3}, \\
(\Box_b)_{22} =& \sum_{l=0}^{n-1} \{(Y_{4l+1} - iY_{4l+2})(-Y_{4l+1} - iY_{4l+2}) \\
& \qquad\qquad\qquad + (Y_{4l+3} + iY_{4l+4})(-Y_{4l+3} + iY_{4l+4}) \\
& + (Y_{4l+3} - iY_{4l+4})(-Y_{4l+3} - iY_{4l+4}) \\
& \qquad\qquad\qquad + (-Y_{4l+1} - iY_{4l+2})(Y_{4l+1} - iY_{4l+2})\} \\
=& \, 2\Delta_b, \\
(\Box_b)_{13} =& (\Box_b)_{31} = 0, \\
(\Box_b)_{33} =& \sum_{l=0}^{n-1} \{(Y_{4l+1} - iY_{4l+2})(-Y_{4l+1} - iY_{4l+2}) \\
& \qquad\qquad\qquad + (Y_{4l+3} + iY_{4l+4})(-Y_{4l+3} + iY_{4l+4})\} \\
=& - \sum_{l=0}^{n-1} \sum_{j=1}^{4} Y_{4l+j}^2 - \sum_{l=0}^{n-1} \{i[Y_{4l+1}, Y_{4l+2}] - i[Y_{4l+3}, Y_{4l+4}]\} = \Delta_b - 8ai\partial_{s_1},
\end{aligned}$$

where $a = \sum_{l=0}^{n-1} a_l$. So (2.11) holds. □

## 3. The fundamental solution of $\Box_b$

Recall that partial Fourier transformation $f \in L^1(\mathscr{H}, \mathbb{C}^3)$ in the variables is defined as

$$\tilde{f}_\tau(y) := \int_{\mathbb{R}^3} e^{-i\tau\cdot s} f(y,s) \mathrm{d}s,$$

where $\tau \cdot s = \sum_{\beta=1}^3 \tau_\beta s_\beta$. Then

$$f(y,s) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\tau\cdot s} \tilde{f}_\tau(y) \mathrm{d}\tau,$$

and

$$\widetilde{\left(\frac{\partial f}{\partial s_\beta}\right)}_\tau (y) = i\tau_\beta \tilde{f}_\tau(y), \quad \widetilde{\left(\frac{\partial^2 f}{\partial s_\beta^2}\right)}_\tau (y) = -\tau_\beta^2 \tilde{f}_\tau(y). \tag{3.1}$$

Note that

$$\begin{aligned}\left(\sum_{\beta=1}^3 \sum_{k=1}^4 B_{kj}^\beta y_{4l+k} \frac{\partial}{\partial s_\beta}\right)^2 &= \sum_{\beta,\beta'=1}^3 \sum_{k,k'=1}^4 B_{kj}^\beta B_{k'j}^{\beta'} y_{4l+k} y_{4l+k'} \frac{\partial}{\partial s_\beta}\frac{\partial}{\partial s_{\beta'}} \\ &= -\sum_{\beta,\beta'=1}^3 \sum_{k,k'=1}^4 \left(B^\beta B^{\beta'}\right)_{kk'} y_{4l+k} y_{4l+k'} \frac{\partial}{\partial s_\beta}\frac{\partial}{\partial s_{\beta'}} \\ &= \sum_{\beta=1}^3 \sum_{k=1}^4 y_{4l+k}^2 \frac{\partial^2}{\partial s_\beta^2},\end{aligned}$$

by $B^\beta B^{\beta'}$ anti-symmetry for $\beta \neq \beta'$ and (2.3). Then we find that

$$\Delta_b = -\sum_{l=0}^{n-1} \sum_{j=1}^4 \left(\partial_{y_{4l+j}}^2 + 4a_l \sum_{\beta=1}^3 \sum_{k=1}^4 B_{kj}^\beta y_{4l+k} \partial_{y_{4l+j}} \partial_{s_\beta} + 4a_l^2 \sum_{\beta=1}^3 \sum_{k=1}^4 y_{4l+k}^2 \partial_{s_\beta}^2\right).$$

Consequently, $\widetilde{(\Delta_b f)}_\tau = \widetilde{\Delta}_{b;\tau} \tilde{f}_\tau$ with $\widetilde{\Delta}_{b;\tau}$ given by

$$\widetilde{\Delta}_{b;\tau} := -\sum_{l=0}^{n-1} \sum_{j=1}^4 \left(\partial_{y_{4l+j}}^2 + 4ia_l \sum_{\beta=1}^3 \sum_{k=1}^4 B_{kj}^\beta y_{4l+k} \tau_\beta \partial_{y_{4l+j}} - 4a_l^2 |\tau|^2 \sum_{j=1}^4 y_{4l+j}^2\right), \tag{3.2}$$

where $|\tau|^2 := \sum_{\beta=1}^3 \tau_\beta^2$, and $\widetilde{(\Box_b f)}_\tau = \widetilde{\Box}_{b;\tau} \tilde{f}_\tau$ with $\widetilde{\Box}_{b;\tau}$ given by

$$\widetilde{\Box}_{b;\tau} := \widetilde{\Delta}_{b;\tau} \cdot \begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} + 8a \begin{pmatrix} -\tau_1 & i\tau_2 + \tau_3 & 0 \\ -i\tau_2 + \tau_3 & 0 & i\tau_2 + \tau_3 \\ 0 & -i\tau_2 + \tau_3 & \tau_1 \end{pmatrix}.$$

If $\tau_2 = \tau_3 = 0$, we can give the fundamental of $\widetilde{\Box}_{b;\tau}$ directly, because in this case $\widetilde{\Box}_{b;\tau}$ is a diagonal matrix. From now we assume $\tau_2^2 + \tau_3^2 \neq 0$. Let us diagonalize

$\widetilde{\Box}_{b;\tau}$. Set

$$\widetilde{\Box}'_{b;\tau} := V_\tau^{-1}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}\widetilde{\Box}_{b;\tau}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}V_\tau, \tag{3.3}$$

where

$$V_\tau = \begin{pmatrix}\frac{\bar\kappa}{\kappa} & -\frac{(\tau_1-|\tau|)^2}{\kappa^2} & -\frac{(\tau_1+|\tau|)^2}{\kappa^2}\\ -\frac{\sqrt{2}\tau_1 i}{\kappa} & -\frac{\sqrt{2}i(\tau_1-|\tau|)}{\kappa} & -\frac{\sqrt{2}i(\tau_1+|\tau|)}{\kappa}\\ 1 & 1 & 1\end{pmatrix}, \quad \kappa=\tau_2+\tau_3 i, \tag{3.4}$$

and the inverse of $V_\tau$ is given by

$$V_\tau^{-1} = \frac{1}{4|\tau|^2}\begin{pmatrix}2\kappa^2 & 2\sqrt{2}\kappa i\tau_1 & 2|\kappa|^2\\ -\kappa^2 & -\sqrt{2}\kappa i(\tau_1+|\tau|) & (\tau_1+|\tau|)^2\\ -\kappa^2 & -\sqrt{2}\kappa i(\tau_1-|\tau|) & (\tau_1-|\tau|)^2\end{pmatrix}. \tag{3.5}$$

We find $V_\tau$ and $V_\tau^{-1}$ by using Matlab. We can see that

$$V_\tau^{-1}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}\widetilde{\Delta}_{b;\tau}\begin{pmatrix}1&&\\&2&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}V_\tau = \widetilde{\Delta}_{b;\tau}\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \tag{3.6}$$

trivially and

$$V_\tau^{-1}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}\begin{pmatrix}-\tau_1 & i\bar\kappa & 0\\ -i\kappa & 0 & i\bar\kappa\\ 0 & -i\kappa & \tau_1\end{pmatrix}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}V_\tau = V_\tau^{-1}\begin{pmatrix}-\tau_1 & \frac{i\bar\kappa}{\sqrt{2}} & 0\\ \frac{-i\kappa}{\sqrt{2}} & 0 & \frac{i\bar\kappa}{\sqrt{2}}\\ 0 & \frac{-i\kappa}{\sqrt{2}} & \tau_1\end{pmatrix}V_\tau$$

$$=\frac{1}{4|\tau|^2}\begin{pmatrix}2\kappa^2 & 2\sqrt{2}\tau_1\kappa i & 2|\kappa|^2\\ -\kappa^2 & -\sqrt{2}\kappa i(\tau_1+|\tau|) & (\tau_1+|\tau|)^2\\ -\kappa^2 & -\sqrt{2}\kappa i(\tau_1-|\tau|) & (\tau_1-|\tau|)^2\end{pmatrix}\begin{pmatrix}-\tau_1 & \frac{i\bar\kappa}{\sqrt{2}} & 0\\ \frac{-i\kappa}{\sqrt{2}} & 0 & \frac{i\bar\kappa}{\sqrt{2}}\\ 0 & \frac{-i\kappa}{\sqrt{2}} & \tau_1\end{pmatrix}V_\tau$$

$$=\frac{1}{4|\tau|}\begin{pmatrix}0 & 0 & 0\\ -\kappa^2 & -\sqrt{2}\kappa(|\tau|+\tau_1)i & (|\tau|+\tau_1)^2\\ \kappa^2 & -\sqrt{2}\kappa(|\tau|-\tau_1)i & -(|\tau|-\tau_1)^2\end{pmatrix}\begin{pmatrix}\frac{\bar\kappa}{\kappa} & -\frac{(\tau_1-|\tau|)^2}{\kappa^2} & -\frac{(\tau_1+|\tau|)^2}{\kappa^2}\\ -\frac{\sqrt{2}\tau_1 i}{\kappa} & -\frac{\sqrt{2}i(\tau_1-|\tau|)}{\kappa} & -\frac{\sqrt{2}i(\tau_1+|\tau|)}{\kappa}\\ 1 & 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}0&&\\&|\tau|&\\&&-|\tau|\end{pmatrix}$$

by direct calculation. Therefore we have

$$\widetilde{\Box}'_{b;\tau} = \begin{pmatrix}\widetilde{\Delta}_{b;\tau}&&\\&\widetilde{\Delta}_{b;\tau}+8a|\tau|&\\&&\widetilde{\Delta}_{b;\tau}-8a|\tau|\end{pmatrix}. \tag{3.7}$$

**Theorem 3.1.** (cf. Theorem 2.8 in [1]) *For*

$$\alpha\notin\Lambda=\left\{-\sum_{j=1}^{4n}\lambda_j(2k_j+1);\boldsymbol{k}=(k_1,\cdots,k_{4n})\in\mathbb{N}^{4n}\right\},$$

*the Hermite operator*

$$H_\alpha = \alpha - \Delta_0 + \sum_{j=1}^{4n} \lambda_j^2 x_j^2, \quad \Delta_0 = \sum_{j=1}^{4n} \partial_{x_j}^2, \quad \lambda_j > 0, \tag{3.8}$$

*has the fundamental solution*

$$K_\alpha(y) = \frac{1}{(2\pi)^{2n}} \int_0^\infty e^{-\alpha t} \left( \prod_{j=1}^{4n} \frac{\lambda_j}{\sinh(2\lambda_j t)} \right)^{\frac{1}{2}} \cdot e^{-\sum_{j=1}^{4n} \left[ \frac{\lambda_j y_j^2}{2\sinh(2\lambda_j t)} + \frac{\lambda_j y_j^2}{2} \tanh(\lambda_j t) \right]} dt.$$

We have the following corollary.

**Corollary 3.1.** *Under the assumption (1.3) for the quaternionic Heisenberg group, the operator $\widetilde{\Box}'_{b;\tau}$ has fundamental solution*

$$\tilde{K}'_\tau(y) = \begin{pmatrix} \tilde{K}'_{1;\tau}(y) & & \\ & \tilde{K}'_{2;\tau}(y) & \\ & & \tilde{K}'_{3;\tau}(y) \end{pmatrix} = \frac{1}{\pi^{2n}} \int_0^\infty \begin{pmatrix} 1 & & \\ & e^{-8a|\tau|t} & \\ & & e^{8a|\tau|t} \end{pmatrix} g(y,\tau;t)\, dt,$$

*where*

$$g(y,\tau;t) = \prod_{l=0}^{n-1} \frac{|a_l\tau|^2}{\sinh^2(4|a_l\tau|t)} e^{-\sum_{l=0}^{n-1}\sum_{j=1}^{4} |a_l\tau| y_{4l+j}^2 \left[ \frac{1}{\sinh(4|a_l\tau|t)} + \tanh(2|a_l\tau|t) \right]}. \tag{3.9}$$

*Proof.* Comparing $\widetilde{\Delta}_{b;\tau}$ in (3.2) with the Hermite operator (3.8), we first note that

$$\sum_{l=0}^{n-1}\sum_{j=1}^{4}\sum_{k=1}^{4} B_{kj}^\beta y_{4l+k} \partial_{y_{4l+j}} \tilde{K}'_\tau(y) = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \tag{3.10}$$

by

$$\sum_{l=0}^{n-1}\sum_{j,k=1}^{4} B_{kj}^\beta y_{4l+k} \partial_{y_{4l+j}} \left( \sum_{l'=0}^{n-1}\sum_{j'=1}^{4} y_{4l'+j'}^2 \right) = 2\sum_{l=0}^{n-1}\sum_{j,k=1}^{4} B_{kj}^\beta y_{4l+k} y_{4l+j} = 0,$$

since $B^\beta$ is anti-symmetric while $y_{4l+k}y_{4l+j}$ is symmetric. Then we can directly check

$$\widetilde{\Box}'_{b;\tau} \tilde{K}'_\tau(y) = \begin{pmatrix} \delta_0 & & \\ & \delta_0 & \\ & & \delta_0 \end{pmatrix},$$

where $\delta_0$ is the Dirac function on $\mathbb{R}^{4n}$. Applying Theorem 3.1 to $\widetilde{\Delta}_{b;\tau} + 8a|\tau|$ with $\lambda_{4l+j} = 2|a_l\tau|$ and $\alpha = 8a|\tau|$, we get $(\widetilde{\Delta}_{b;\tau} + 8a|\tau|)\tilde{K}'_{2;\tau} = \delta_0$, when

$$8a \notin \left\{ -2\sum_{l=0}^{n-1}\sum_{j=1}^{4} |a_l|(2k_{4l+j}+1); \mathbf{k} = (k_1, \cdots, k_{4n}) \in \mathbb{N}^{4n} \right\}. \tag{3.11}$$

As every element of $\mathbf{k}$ is nonnegative, $8a = -2\sum_{l=0}^{n-1}\sum_{j=1}^{4}|a_l|(2k_{4l+j}+1)$ may only happen when $\mathbf{k} = \mathbf{0}$. It reduces to

$$a \neq -\sum_{l=0}^{n-1}|a_l|.$$

Similarly, applying Theorem 3.1 with $\lambda_{4l+j} = 2|a_l\tau|$, $\alpha = 0$ or $\alpha = -8a|\tau|$, we get $\widetilde{\Delta}_{b;\tau}\tilde{K}'_{1;\tau} = \delta_0$ for any $a_l$'s and $(\widetilde{\Delta}_{b;\tau} - 8a|\tau|)\tilde{K}'_{3;\tau} = \delta_0$ when $a \neq \sum_{l=0}^{n-1}|a_l|$, respectively. The corollary is proved. □

By direct calculation, we have the following proposition.

**Proposition 3.1.** *Under the assumption (1.3) for the quaternionic Heisenberg group, the fundamental solution of* $\widetilde{\Box}_{b;\tau}$ *is*

$$\tilde{K}_\tau(y) = \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{2}} & \\ & & 1 \end{pmatrix} V_\tau \tilde{K}'_\tau(y) V_\tau^{-1} \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{2}} & \\ & & 1 \end{pmatrix}. \tag{3.12}$$

*Proof.* By (3.3) and Corollary 3.1, we have

$$\begin{aligned}
&\widetilde{\Box}_{b;\tau}\tilde{K}_\tau(y)\\
&= \begin{pmatrix} 1 & & \\ & \sqrt{2} & \\ & & 1 \end{pmatrix} V_\tau \widetilde{\Box}'_{b;\tau} V_\tau^{-1} \begin{pmatrix} 1 & & \\ & \sqrt{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{2}} & \\ & & 1 \end{pmatrix} V_\tau \tilde{K}'_\tau(y) V_\tau^{-1} \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{2}} & \\ & & 1 \end{pmatrix}\\
&= \begin{pmatrix} 1 & & \\ & \sqrt{2} & \\ & & 1 \end{pmatrix} V_\tau \begin{pmatrix} \delta_0 & & \\ & \delta_0 & \\ & & \delta_0 \end{pmatrix} V_\tau^{-1} \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{2}} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \delta_0 & & \\ & \delta_0 & \\ & & \delta_0 \end{pmatrix}.
\end{aligned}$$

The proposition is proved. □

Set

$$\dot{\tau}_\beta := \frac{\tau_\beta}{|\tau|},\ \beta = 1,2,3, \quad \dot{\kappa} := \frac{\kappa}{|\tau|}.$$

**Corollary 3.2.** *Under the assumption (1.3) for the quaternionic Heisenberg group, the fundamental solution of* $\widetilde{\Box}_{b;\tau}$ *is*

$$\begin{aligned}
&\tilde{K}_\tau(y)\\
&= \begin{pmatrix} \frac{2\tilde{K}'_{1;\tau}|\dot{\kappa}|^2+\tilde{K}'_{2;\tau}(1-\dot{\tau}_1)^2+\tilde{K}'_{3;\tau}(1+\dot{\tau}_1)^2}{4} & \frac{\bar{\dot{\kappa}}iA}{4} & \frac{\bar{\dot{\kappa}}^2C}{4} \\ -\frac{\dot{\kappa}iA}{4} & \frac{2\tilde{K}'_{1;\tau}\dot{\tau}_1^2+(\tilde{K}'_{2;\tau}+\tilde{K}'_{3;\tau})|\dot{\kappa}|^2}{2} & \frac{\bar{\dot{\kappa}}iB}{4} \\ \frac{\dot{\kappa}^2C}{4} & -\frac{\dot{\kappa}iB}{4} & \frac{2\tilde{K}'_{1;\tau}|\dot{\kappa}|^2+\tilde{K}'_{2;\tau}(1+\dot{\tau}_1^2)+\tilde{K}'_{3;\tau}(1-\dot{\tau}_1)^2}{4} \end{pmatrix},
\end{aligned} \tag{3.13}$$

*where*

$$\begin{aligned}
A &= 2\tilde{K}'_{1;\tau}\dot{\tau}_1 + \tilde{K}'_{2;\tau}(1-\dot{\tau}_1) - \tilde{K}'_{3;\tau}(\dot{\tau}_1+1),\\
B &= -2\tilde{K}'_{1;\tau}\dot{\tau}_1 + \tilde{K}'_{2;\tau}(1+\dot{\tau}_1) - \tilde{K}'_{3;\tau}(1-\dot{\tau}_1),\\
C &= 2\tilde{K}'_{1;\tau} - \tilde{K}'_{2;\tau} - \tilde{K}'_{3;\tau}.
\end{aligned}$$

*Proof.* By Proposition 3.1, we have

$$\begin{aligned}&\tilde{K}_\tau(y)\\&=\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}V_\tau\begin{pmatrix}\tilde{K}'_{1;\tau}(y)&&\\&\tilde{K}'_{2;\tau}(y)&\\&&\tilde{K}'_{3;\tau}(y)\end{pmatrix}V_\tau^{-1}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}\end{aligned}$$

$$=\frac{1}{4|\tau|^2}\begin{pmatrix}\frac{\bar\kappa}{\kappa}&-\frac{(\tau_1-|\tau|)^2}{\kappa^2}&-\frac{(\tau_1+|\tau|)^2}{\kappa^2}\\-\frac{\tau_1 i}{\kappa}&-\frac{i(\tau_1-|\tau|)}{\kappa}&-\frac{i(\tau_1+|\tau|)}{\kappa}\\1&1&1\end{pmatrix}\begin{pmatrix}\tilde{K}'_{1;\tau}(y)&&\\&\tilde{K}'_{2;\tau}(y)&\\&&\tilde{K}'_{3;\tau}(y)\end{pmatrix}\begin{pmatrix}2\kappa^2&2\tau_1\kappa i&2|\kappa|^2\\-\kappa^2&-\kappa(\tau_1+|\tau|)i&(\tau_1+|\tau|)^2\\-\kappa^2&-\kappa(\tau_1-|\tau|)i&(\tau_1-|\tau|)^2\end{pmatrix},$$

where $V_\tau$ is defined in (3.4). Then by direct calculation we get (3.13). $\Box$

At last we get the following theorem.

**Theorem 3.2.** *Under the assumption (1.3) for the quaternionic Heisenberg group, the fundamental solution of $\Box_b$ is*

$$K(y,s)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{is\cdot\tau}\tilde{K}_\tau(y)\,d\tau. \tag{3.14}$$

For a function $f$ on $\mathscr{H}$, we say $f$ *homogeneous* of degree $\lambda$ if $f(ry,r^2s)=r^\lambda f(y,s)$ for all $r>0$.

**Corollary 3.3.** *$K(y,s)$ is homogeneous of degree $-Q+2$, where $Q=4n+6$ is the homogeneous dimension of $\mathscr{H}$.*

*Proof.* In terms of the polar coordinates $\tau=|\tau|\dot\tau=|\tau|(\dot\tau_1,\dot\tau_2,\dot\tau_3)$ for the $\tau$-variable, we have

$$\begin{aligned}K(y,s)=&\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{is\cdot\tau}U_\tau\begin{pmatrix}\tilde{K}'_{1;\tau}(y)&&\\&\tilde{K}'_{2;\tau}(y)&\\&&\tilde{K}'_{3;\tau}(y)\end{pmatrix}U'_\tau\mathrm{d}\tau\\=&\frac{1}{8\pi^{2n+3}}\int_{S^2}\mathrm{d}\dot\tau\int_0^\infty\mathrm{d}|\tau|\int_0^\infty\mathrm{d}t|\tau|^2e^{is\cdot\dot\tau|\tau|}U_\tau\begin{pmatrix}1&&\\&e^{-8a|\tau|\dot\tau\cdot t}&\\&&e^{8a|\tau|\dot\tau\cdot t}\end{pmatrix}g(y,\tau;t)U'_\tau,\end{aligned}$$

by Theorem 3.2 and Proposition 3.1, where

$$U_\tau=\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix}V_\tau,\quad U'_\tau=V_\tau^{-1}\begin{pmatrix}1&&\\&\frac{1}{\sqrt{2}}&\\&&1\end{pmatrix},$$

and $g(y,\tau;t)$ is given by (3.9). It follows from (3.4)-(3.5) and (3.9) that

$$U_{r^2\tau}=U_\tau,\ U'_{r^2\tau}=U'_\tau\quad\text{and}\quad g(ry,\tau;t)=\frac{1}{r^{4n}}g\left(y,r^2\tau;\frac{t}{r^2}\right).$$

We find that

$$\begin{aligned} &K(ry, r^2 s)\\ =&\frac{1}{r^{4n}}\frac{1}{(2\pi)^3}\int_{S^2}\mathrm{d}\dot{\tau}\int_0^\infty \mathrm{d}|\tau|\int_0^\infty \mathrm{d}t|\tau|^2 e^{ir^2 s\cdot\dot{\tau}|\tau|}U_\tau\begin{pmatrix}1 & & \\ & e^{8a|\tau|\dot{\tau}\cdot t} & \\ & & e^{-8a|\tau|\dot{\tau}\cdot t}\end{pmatrix} g\left(y, r^2\tau; \frac{t}{r^2}\right)U'_\tau\\ =&\frac{1}{r^{Q-2}}K(y,s), \end{aligned}$$

by changing variables $r^2\tau \to \tau$, $\frac{t}{r^2} \to t$. The corollary is proved. $\square$

Recall the convolution of the functions $f \in L^1(\mathscr{H}, \mathbb{C}^{3\times 3})$ and $g \in L^1(\mathscr{H}, \mathbb{C}^3)$ on $\mathscr{H}$ is defined as

$$f * g(y,s) = \int f(x,t)g((x,t)^{-1}(y,s))\mathrm{d}x\mathrm{d}t = \int f((y,s)(x,t)^{-1})g(x,t)\mathrm{d}x\mathrm{d}t.$$

If we set $\tilde{g}(y,s) = g(-y,-s)$, then

$$\int \langle (f*g)(y,s), h(y,s)\rangle_{\mathbb{C}^3}\mathrm{d}y\mathrm{d}s = \int \left\langle g(y,s), \left(\overline{\tilde{f}}^t * h\right)(y,s)\right\rangle_{\mathbb{C}^3}\mathrm{d}y\mathrm{d}s, \tag{3.15}$$

whenever both side make sense. For $f \in C_0^\infty(\mathscr{H}, \mathbb{C}^3)$, define

$$(Tf)(y,s) = (K*f)(y,s).$$

As $K(y,s)$ is homogeneous of degree $-4n-4$, we can prove the following proposition for the quaternionic Heisenberg group exactly in the same way as Proposition 9.3 in Folland-Stein [3] for the Heisenberg group. We omit the detail.

**Proposition 3.2.** *Let $f \in C_0^\infty(\mathscr{H}, \mathbb{C}^3)$. Then the mapping $f \to Tf = K*f$ extends to a bounded mapping from $L^p(\mathscr{H}, \mathbb{C}^3)$ to $L^q(\mathscr{H}, \mathbb{C}^3)$, where $\frac{1}{q} = \frac{1}{p} - \frac{2}{Q}$ provided $1 < p < q < \infty$.*

**Proposition 3.3.** *If $f \in C_0^\infty(\mathscr{H}, \mathbb{C}^3)$, $\Box_b Tf = T\Box_b f = f$.*

*Proof.* It is obvious that $\Box_b Tf = \Box_b K * f = \delta * f = f$. On the other hand, for any $g \in C_0^\infty(\mathscr{H}, \mathbb{C}^3)$,

$$\begin{aligned} \int \langle g(y,s), f(y,s)\rangle_{\mathbb{C}^3}\mathrm{d}y\mathrm{d}s &= \int \langle \Box_b Tg(y,s), f(y,s)\rangle_{\mathbb{C}^3}\mathrm{d}y\mathrm{d}s\\ &= \int \langle Tg(y,s), \Box_b f(y,s)\rangle_{\mathbb{C}^3}\mathrm{d}y\mathrm{d}s\\ &= \int \left\langle g(y,s), \left(\overline{\tilde{K}}^t * \Box_b f\right)(y,s)\right\rangle_{\mathbb{C}^3}\mathrm{d}y\mathrm{d}s\\ &= \int \langle g(y,s), T\Box_b f(y,s)\rangle_{\mathbb{C}^3}\,\mathrm{d}y\mathrm{d}s, \end{aligned}$$

where (3.15) is used in the third identity. The last identity holds because

$$\overline{\tilde{K}(y,s)}^t = \overline{K(-y,-s)}^t = K(y,s)$$

by (3.14) and unitariness of the matrix $\tilde{K}_\tau(y)$ in (3.13). Then $T\Box_b f = f$. $\square$

See Proposition 7.1 in Folland-Stein [3] for this proposition in the CR case.

*Proof of Theorem 1.1.* Obviously we have $\mathcal{A}_p(\mathscr{H}, \mathbb{C}^3) \subset \ker \Box_b$. We only need to prove $\ker \Box_b = \{0\}$. For any $f \in L^p$ and $f \in \ker \Box_b$, we have

$$f = T\Box_b f = 0 \quad \text{in} \quad L^q,$$

where $\frac{1}{q} = \frac{1}{p} - \frac{2}{Q}$ for $1 < p < q < \infty$. The theorem follows.

**Remark 3.1.** *The case $k = 1$ is much easier. For the case $k = 3$, we have*

$$\Box_b^{(3)} = \Delta_b \cdot \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} + 8a \begin{pmatrix} i\partial_{s_1} & \partial_{s_2} - i\partial_{s_3} & 0 & 0 \\ -\partial_{s_2} - i\partial_{s_3} & 0 & \partial_{s_2} - i\partial_{s_3} & 0 \\ 0 & -\partial_{s_2} - i\partial_{s_3} & 0 & \partial_{s_2} - i\partial_{s_3} \\ 0 & 0 & -\partial_{s_2} - i\partial_{s_3} & -i\partial_{s_1} \end{pmatrix},$$

*and then*

$$\widetilde{\Box}_{b;\tau}^{(3)} = \widetilde{\Delta}_{b;\tau} \cdot \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} + 8a \begin{pmatrix} -\tau_1 & i\tau_2 + \tau_3 & 0 & 0 \\ -i\tau_2 + \tau_3 & 0 & i\tau_2 + \tau_3 & 0 \\ 0 & -i\tau_2 + \tau_3 & 0 & i\tau_2 + \tau_3 \\ 0 & 0 & -i\tau_2 + \tau_3 & \tau_1 \end{pmatrix}.$$

*The calculation is similar to the case $k = 2$. We can also diagonalize $\widetilde{\Box}_{b;\tau}^{(3)}$ by multiplying*

$$V_\tau^{-1} \begin{pmatrix} 1 & & & \\ & \frac{1}{\sqrt{2}} & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & & \\ & \frac{1}{\sqrt{2}} & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & 1 \end{pmatrix} V_\tau$$

*to its left and right, respectively. We can find*

$$V_\tau = \begin{pmatrix} \frac{(|\tau|-\tau_1)^3 i}{\kappa^3} & \frac{(|\tau|+\tau_1)^3 i}{\kappa^3} & -\frac{i\bar{\kappa}|\kappa|}{\kappa^2} & \frac{i\bar{\kappa}|\kappa|}{\kappa^2} \\ -\frac{\sqrt{2}(\tau_1-|\tau|)^2}{\kappa^2} & -\frac{\sqrt{2}(\tau_1+|\tau|)^2}{\kappa^2} & \frac{|\kappa|(|\tau|-2\tau_1)}{\sqrt{2}\kappa^2} & \frac{|\kappa|(|\tau|+2\tau_1)}{\sqrt{2}\kappa^2} \\ -\frac{\sqrt{2}i(\tau_1-|\tau|)}{\kappa} & -\frac{\sqrt{2}i(\tau_1+|\tau|)}{\kappa} & -\frac{(|\tau|+2\tau_1)i}{\sqrt{2}\kappa} & \frac{(|\tau|-2\tau_1)i}{\sqrt{2}\kappa} \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \kappa = \tau_2 + \tau_3 i,$$

*by using Matlab. Similarly we have Theorem 1.1 and 3.2 for $k = 3$. It seems that we can give these theorems for any fixed $k = 4, 5, \ldots$, by using Matlab. But we can not write down general form of $V_\tau$ and $V_\tau^{-1}$.*

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