Applied Analysis and Optimization
Volume 1, Number 1, 2017, 99–112

Yokohama Publishers
ISSN 2189-1664 Online Journal

FIXED POINTS AND COUPLED FIXED POINTS FOR MULTI-VALUED (φ, ψ)-CONTRACTIONS IN b-METRIC SPACES

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Abstract. In this paper, we will study the coupled fixed point problem for multi-valued operators satisfying a nonlinear contraction condition. The approach is based on a fixed point theorem for multi-valued operators in a complete b-metric space.

1. Introduction

The context of the results given in this paper is that of a complete b-metric space.

Definition 1.1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if the following axioms are satisfied:

i) if $x, y \in X$, then $d(x, y) = 0 \Leftrightarrow x = y$;
ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
iii) $d(x, z) \leq s\,[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

A pair (X, d) with the above properties is called a b-metric space.

For some examples of b-metric spaces see [1], [4], [7].

Let (X, d) be a b-metric space and $\mathcal{P}(X)$ be the set of all subset of X.

In this paper, we will use the following notations:

$$P(X) = \{Y \in \mathcal{P}(X) /\ Y \neq \emptyset\};\ P_{cl}(X) = \{Y \in \mathcal{P}(X) /\ Y \text{ is closed}\}.$$

If $T : X \to P(X)$ is a multi-valued operator, then $x \in X$ is called fixed point for $T \Leftrightarrow x \in T(x)$.

$$Fix(T) = \{x \in X /\ x \in T(x)\} \text{ is the fixed point set of } T.$$

and $SFix(T) = \{x \in X /\ T(x) = \{x\}\}$ is the set of all strict fixed points of T.

Moreover, we will denote by

$$Graph(T) = \{(x, y) \in X \times X /\ y \in T(x)\} \text{ the graph of } T.$$

Let (X, d) be a b-metric space with constant $s \geq 1$ and $Z = X \times X$. Then, the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by $\tilde{d}\,((x, y), (u, v)) = d(x, u) + d(y, v)$ for all $(x, y), (u, v) \in Z$ is a b-metric on Z with the same constant $s \geq 1$ and if (X, d) is a complete b-metric space, then $(Z, \tilde{d})$ is a complete b-metric space, too.

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. b-metric space, multi-valued operator, fixed point, coupled fixed point, Hausdorff-Pompeiu metric, data dependence, well-posedness, Ulam-Hyers stability.

Moreover, for $x,\ y \in X,\ A, B, U, V \in P(X)$ we have:

$$D_{\tilde{d}}((x,y), U \times V) = D_d(x,U) + D_d(y,V);$$

$$\rho_{\tilde{d}}(A \times B, U \times V) = \rho_d(A,U) + \rho_d(B,V);$$

$$H_{\tilde{d}}(A \times B, U \times V) \leq H_d(A,U) + H_d(B,V),$$

where the following notations are used:

(1) for the gap functional generated by d "D_d":

$$D_d : P(X) \times P(X) \to \mathbb{R}_+,\ D_d(A,B) = \inf\{d(a,b)/\ a \in A,\ b \in B\};$$

(2) for the excess generalized functional "ρ_d":

$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \rho_d(A,B) = \sup\{D_d(a,B)/\ a \in A\};$$

(3) for the Hausdorff-Pompeiu generalized functional "H_d":

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A,B) = \max\{\rho_d(A,B), \rho_d(B,A)\}.$$

Additionally, by the properties of the gap functional D_d, if $(x,y) \in X \times X$ and $A, B \in P_{cl}(X)$, then

$$D_{\tilde{d}}((x,y), U \times V) = 0 \Leftrightarrow (x,y) \in U \times V.$$

Definition 1.2. Let $(X, \leq)$ be a partially ordered set. Then, the partial order "$\leq$" induces on the product space $X \times X$ the following partial order relation:

$$\text{for } (x,y), (u,v) \in X \times X\ (x,y) \leq_p (u,v) \Leftrightarrow x \leq u,\ y \geq v.$$

Definition 1.3. Let X be a nonempty set, let "$\leq$" be a partial order on X and d be a b-metric on X with constant $s \geq 1$. Then the triple $(X, \leq, d)$ is called an ordered b-metric space if:

(i) "$\leq$" is a partially order on X;
(ii) d is a b-metric on X with constant $s \geq 1$;
(iii) if $(x_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence in X and $\lim\limits_{n\to\infty} x_n = x^*$ then $x_n \leq x^*$ for all $n \in \mathbb{N}$;
(iv) if $(y_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence in X and $\lim\limits_{n\to\infty} y_n = y^*$ then $y_n \geq y^*$ for all $n \in \mathbb{N}$.

Definition 1.4. Let $(X, \leq)$ be a partially ordered set and $A, B \in P(X)$. We will denote:

a) $A \leq_{st} B \Leftrightarrow \forall a \in A, \forall b \in B$ we have $a \leq b$;
b) $A \leq_{wk} B \Leftrightarrow \forall a \in A, \exists b \in B$ such that $a \leq b$.

Definition 1.5. Let $(X, \leq)$ be a partially ordered set and $T : X \to P(X)$ be a multi-valued operator. We say that T is strong increasing (respectively strong decreasing) on X if for every $x, y \in X$ with $x \leq y$ we have that $T(x) \leq_{st} T(y)$ (respectively $T(x) \geq_{st} T(y)$).

Let (X,d) be a metric space and $T : X \times X \to P(X)$ be a multi-valued operator. Following [6] (where the single-valued case is treated), by definition, a coupled fixed point problem for T means to find a pair $(x^*, y^*) \in X \times X$ satisfying

$$(P) \quad \begin{cases} x^* \in T(x^*, y^*) \\ y^* \in T(y^*, x^*) \end{cases}$$

The purpose of this paper, is to study the coupled fixed point problem for multi-valued operators satisfying a nonlinear contraction condition. The approach is based on some fixed point theorems for multi-valued operators in complete b-metric space. Several properties of the solution set of the coupled fixed point problem will be also discussed. Our results extend and complement some theorems given in [2], [8], [10], [11], [12].

2. Fixed point theorems for (φ, ψ)-contractions

We recall first the following auxiliary result.

Lemma 2.1. *Let (X, d) be a b-metric space and $\epsilon > 0$. Let $A, B \in P(X)$. Then $\forall a \in A, \exists b \in B$ such that*

$$d(a, b) \leq H(A, B) + \epsilon$$

Let Φ denote the set of all function $\varphi : [0, \infty) \to [0, \infty)$ satisfying:

(i_φ) φ is continuous and (strictly) increasing;
(ii_φ) $\varphi(t) < t$ for all $t > 0$;
(iii_φ) $\varphi(a + b) \leq \varphi(a) + b,\ \forall a, b \in [0, \infty)$;
(iv_φ) $\varphi(st) \leq s\varphi(t),$ (where $s \geq 1$), $\forall t \in [0, \infty)$.

We denote by Ψ the set of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy:

(i_ψ) $\lim\limits_{t \to r} \psi(t) > 0$ for all $r > 0$;
(ii_ψ) $\lim\limits_{t \to 0_+} \psi(t) = 0$.

Theorem 2.2. *Let $(X, \leq, d)$ be a complete ordered b-metric space with constant $s \geq 1$. Let $T : X \to P_{cl}(X)$ be a multivalued operator strong increasing with respect to " $\leq$ ". Suppose that:*

(i) *there exist two functionals $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $(x, y) \in X \times X$ with $x \leq y$:*

$$\varphi(H_d(T(x), T(y))) \leq \varphi(d(x, y)) - \psi(d(x, y));$$

(ii) *there exists an element $x_0 \in X$ such that $x_0 \leq_{wk} T(x_0)$.*

Then $Fix(T) \neq \emptyset$ and there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X of successive approximation of T starting from $x_0 \in X$ which converges to a fixed point of T.

Proof. Let $x_0 \in X$ such that $x_0 \leq_{wk} T(x_0)$. Then, there exists $x_1 \in T(x_0)$ such that $x_0 \leq x_1$.
Suppose $x_0 \neq x_1$. Otherwise $x_0 \in T(x_0) \Rightarrow Fix(T) \neq \emptyset$.
Let $\tilde{\varepsilon} > 0$.
Using Lemma 2.1 for any $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \leq H_d(T(x_0), T(x_1)) + \tilde{\varepsilon}$

$$\Rightarrow \varphi(d(x_1, x_2)) \leq \varphi(H_d(T(x_0), T(x_1)) + \tilde{\varepsilon}) \leq \varphi(H_d(T(x_0), T(x_1))) + \tilde{\varepsilon}.$$

Since $d(x_0, x_1) > 0 \Rightarrow \varphi(d(x_0, x_1)) > 0$. By our hypothesis, we have

$$\varphi(H_d(T(x_0), T(x_1))) \leq \varphi(d(x_0, x_1)) - \psi(d(x_0, x_1)) < \varphi(d(x_0, x_1)).$$

We choose

$$\tilde{\varepsilon} := \varphi(d(x_0, x_1)) - \varphi(H_d(T(x_0), T(x_1))) > 0,$$

and we get

$$\varphi(d(x_1, x_2)) \leq \varphi(d(x_0, x_1)).$$

Since φ is increasing, we get that $d(x_1, x_2) \leq d(x_0, x_1)$.
Since $x_1 \in T(x_0), x_2 \in T(x_1), x_0 \leq x_1$ and because T is strong increasing $\Rightarrow x_1 \leq x_2$.
Suppose $x_1 \neq x_2$. Otherwise $x_1 \in Fix(T) \Rightarrow Fix(T) \neq \emptyset$.
Using Lemma 2.1 for any $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that $d(x_2, x_3) \leq H_d(T(x_1), T(x_2)) + \tilde{\varepsilon}$

$$\Rightarrow \varphi(d(x_2, x_3)) \leq \varphi(H_d(T(x_1), T(x_2)) + \tilde{\varepsilon}) \leq \varphi(H_d(T(x_1), T(x_2))) + \tilde{\varepsilon}.$$

Since $d(x_1, x_2) > 0$ we get $\varphi(d(x_1, x_2)) > 0$.
Thus

$$\varphi(H_d(T(x_1), T(x_2))) \leq \varphi(d(x_1, x_2)) - \psi(d(x_1, x_2)) < \varphi(d(x_1, x_2)).$$

We choose

$$\tilde{\varepsilon} := \varphi(d(x_1, x_2)) - \varphi(H_d(T(x_1), T(x_2))) > 0,$$

and we get

$$\varphi(d(x_2, x_3)) \leq \varphi(d(x_1, x_2)).$$

By the monotonicity of φ, we get that $\varphi(d(x_2, x_3)) \leq \varphi(d(x_1, x_2)) \leq \varphi(d(x_0, x_1))$.
By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in X with the following properties:

(a) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;
(b) $x_n \leq x_{n+1}$, for all $n \in \mathbb{N}$;
(c) $\varphi(d(x_n, x_{n+1})) \leq \varphi(d(x_{n-1}, x_n))$, for all $n \in \mathbb{N}$, φ is increasing;
(c') $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \forall n \in \mathbb{N}$;
(d) $\varphi(d(x_n, x_{n+1})) \leq \varphi(d(x_{n-1}, x_n)) - \psi(d(x_{n-1}, x_n)) + \tilde{\varepsilon},\ \forall \tilde{\varepsilon} > 0$.

Then by (c') we have: $0 \leq \delta_{n+1} := d(x_n, x_{n+1})$ is decreasing.
Thus $\lim\limits_{n\to\infty} \delta_n = \delta \geq 0$.

We show that $\delta = 0$ (by contradiction).
Assume the contrary, that is $\delta > 0$. Then by letting $n \to \infty$ in (d) we get:

$$\varphi(\delta) = \lim_{n\to\infty} \varphi(\delta_{n+1}) \leq \lim_{n\to\infty} \varphi(\delta_n) - \lim_{n\to\infty} \psi(\delta_n) + \tilde{\varepsilon} < \varphi(\delta) - \lim_{n\to\infty} \psi(\delta_n) + \tilde{\varepsilon}.$$

We choose $\tilde{\varepsilon} < \lim\limits_{n\to\infty} \psi(\delta_n) > 0$. Then we obtain: $0 < -\lim\limits_{n\to\infty} \psi(\delta_n) + \tilde{\varepsilon} < 0$ gives a contradiction. Follows that $\delta = 0$. Further, we proof by contradiction that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
Suppose that $(x_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence on $X \Rightarrow \exists \varepsilon > 0$ for which we can find two sub-sequences $(x_{n(k)})$ and $(x_{m(k)})$ of $(x_n)_{n\in\mathbb{N}}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) \geq s \cdot \varepsilon,\ k = 1, 2, ... \tag{2.1}$$

We can choose $n(k)$ to be the smallest integer with property $n(k) > m(k) \geq k$ and satisfying (2.1), follows that $d(x_{n(k)-1}, x_{m(k)}) < \varepsilon$.

By triangle inequality we have:

$$s\varepsilon \le r_k = d(x_{n(k)}, x_{m(k)}) \le s[d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)})] <$$
$$< s \cdot d(x_{n(k)}, x_{n(k)-1}) + s \cdot \varepsilon.$$

Letting $k \to \infty$ we get $\lim_{k\to\infty} r_k = s \cdot \varepsilon$.

Since $n(k) > m(k)$ we have, using the property (b) that $x_{n(k)} \ge x_{m(k)}$ or $x_{m(k)} \le x_{n(k)}$, we put $x = x_{m(k)}$ and $y = x_{n(k)}$ in $(i) \Rightarrow$

$$\varphi(H_d(T(x_{m(k)}), T(x_{n(k)}))) \le \varphi(d(x_{m(k)}, x_{n(k)})) - \psi(d(x_{m(k)}, x_{n(k)})).$$

We have:

$$\Rightarrow \varphi\left(d(x_{m(k)+1}, x_{n(k)+1})\right) \le \varphi\left(H_d\left(T(x_{m(k)}), T(x_{n(k)})\right) + \tilde{\varepsilon}\right)$$
$$\le \varphi\left(H_d\left(T(x_{m(k)}), T(x_{n(k)})\right)\right) + \tilde{\varepsilon} \le \varphi\left(d(x_{m(k)}, x_{n(k)})\right) - \psi\left(d(x_{m(k)}, x_{n(k)})\right) + \tilde{\varepsilon}.$$

follows that

$$\varphi(r_{k+1}) \le \varphi(r_k) - \psi(r_k) + \tilde{\varepsilon} \text{ and letting } k \to \infty \Rightarrow \varphi(s\varepsilon) \le \varphi(s\varepsilon) - \lim_{k\to\infty} \psi(r_k) + \tilde{\varepsilon}$$
$$\Rightarrow 0 \le -\lim_{k\to\infty} \psi(r_k) + \tilde{\varepsilon} \text{ and if we chose } \tilde{\varepsilon} < \lim_{k\to\infty} \psi(r_k) > 0$$

$\Rightarrow 0 < 0$ which gives a contradiction.

Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete b-metric space $(X, d) \Rightarrow \exists x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, we show that $x^* \in T(x^*)$

$$\begin{aligned}
D_d(x^*, T(x^*)) &\le s[d(x^*, x_{n+1}) + D_d(x_{n+1}, T(x^*))] \\
&\le s \cdot d(x^*, x_{n+1}) + s \cdot H_d(T(x_n), T(x^*)) \\
&\Rightarrow D_d(x^*, T(x^*)) - s \cdot d(x^*, x_{n+1}) \le s \cdot H_d(T(x_n), T(x^*)) \\
&\Rightarrow \varphi\left(D_d(x^*, T(x^*)) - s \cdot d(x^*, x_{n+1})\right) \le s \cdot \varphi\left(H_d(T(x_n), T(x^*))\right) \\
&\underset{(i)}{\overset{x_n \le x^*}{\le}} s\left[\varphi(d(x_n, x^*)) - \psi(d(x_n, x^*))\right].
\end{aligned}$$

Letting $n \to \infty$ we obtain that:

$$\varphi\left(D_d(x^*, T(x^*)) - 0\right) \le s\left[\varphi(0) - 0\right].$$

On the other hand $\varphi(t) = 0 \Leftrightarrow t = 0$ and T has closed values, so $x^* \in T(x^*) \Rightarrow Fix(T) \ne \emptyset$. □

Our next result is a fixed point theorem for a multi-valued operator satisfying a (φ, ψ)-contraction type condition on the whole space.

Theorem 2.3. *Let (X, d) be a complete b-metric space with constant $s \ge 1$ and $T : X \to P_{cl}(X)$ a multi-valued operator for which there exist two functionals $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $(x, y) \in X \times X$ we have:*

$$\varphi\left(H_d(T(x), T(y))\right) \le \varphi(d(x, y)) - \psi(d(x, y)).$$

Then

(a) $Fix(T) \neq \emptyset$ *and there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *of successive approximations of* T *starting from any* $(x_0, x_1) \in Graph(T)$ *which converges to a fixed point* x^* *of* T.

(b) *If additionally* $SFix(T) \neq \emptyset$ *and* $\psi \in \Psi$ *is a continuous mapping then* $Fix(T) = SFix(T) = \{x^*\}$.

Proof. (a) is the same with the proof of Theorem 2.2.

(b) Let $y^* \in SFix(T)$. We consider $y \in SFix(T)$ such that $y \neq y^*$. Then $d(y, y^*) = H_d(T(y), T(y^*)) \Rightarrow$

$$\Rightarrow \varphi\big(d(y, y^*)\big) = \varphi\big(H_d(T(y), T(y^*))\big) \leq \varphi(d(y, y^*)) - \psi(d(y, y^*)).$$

Since $y \neq y^* \Rightarrow \varphi(d(y, y^*)) > 0$ and since ψ is continuous mapping that means $\psi(d(y, y^*)) > 0$.

Thus $0 \leq -\psi(d(y, y^*))$ and we have a contradiction

$$\Rightarrow y = y^* \Rightarrow SFix(T) = \{y^*\}.$$

Let $x^* \in Fix(T)$ with $x^* \neq y^* \Rightarrow d(x^*, y^*) > 0 \Rightarrow \varphi(d(x^*, y^*)) > 0$.

We have $d(x^*, y^*) = D_d(x^*, T(y^*)) \leq H_d(T(x^*), T(y^*))$

$$\Rightarrow \varphi\big(d(x^*, y^*)\big) \leq \varphi\big(H_d(T(x^*), T(y^*))\big) \leq \varphi(d(x^*, y^*)) - \psi(d(x^*, y^*))$$

$$\Rightarrow 0 \leq -\psi(d(x^*, y^*)), \text{ which is a contradiction.}$$

Finally we have $SFix(T) = Fix(T) = \{x^*\}$. □

For related results see [3], [9].

3. Coupled fixed point theorems for multi-valued (φ, ψ) - contractions

We have the following useful definition.

Definition 3.1. Let $(X, \leq)$ be a partially ordered set and $G : X \times X \to P(X)$. We say that G has the strict mixed monotone property with respect to the partial order " $\leq$ " if the following implications hold:

a) $x_0 \leq x_1 \Rightarrow G(x_0, y) \leq_{st} G(x_1, y), \ \forall y \in X$

b) $y_0 \geq y_1 \Rightarrow G(x, y_0) \leq_{st} G(x, y_1), \ \forall x \in X$.

We recall now the following theorem, which was the starting point in the coupled fixed point theory for single-valued operators.

Theorem 3.2 (Bhaskar and Lakshmikantham)**.** *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *be a continuous mapping having the mixed monotone property on* X. *Assume that there exists a constant* $k \in [0, 1)$ *with*

$$d\big(F(x, y), F(u, v)\big) \leq \frac{k}{2}[d(x, u) + d(y, v)] \text{ for each } x \leq u, y \geq v \tag{3.1}$$

If there exist $x_0, y_0 \in X$ *such that*

$$x_0 \leq F(x_0, y_0), \ y_0 \geq F(y_0, x_0)$$

then there exist $x^*, y^* \in X$ *such that*

$$\begin{cases} x^* = F(x^*, y^*) \\ y^* = F(y^*, x^*) \end{cases}.$$

Remark 3.3. The hypothesis $F : X \times X \to X$ is a continuous mapping can be replaced by **Assumption 1**: X has the property that:

(a) if a non-decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to x then $x_n \leq x$, for all $n \in \mathbb{N}$;
(b) if a non-increasing sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to x then $x_n \geq x$, for all $n \in \mathbb{N}$.

Later on, Luong and Thuan use a more general contraction type condition:

$$\varphi\big(d(F(x,y), F(u,v))\big) \leq \frac{1}{2}\varphi(d(x,u) + d(y,v)) - \psi(d(x,u) + d(y,v)), \tag{3.2}$$

where $\varphi,\ \psi : [0,\infty) \to [0,\infty)$ are functions satisfying some appropriate conditions and $(x,y), (u,v) \in X \times X$ with $x \leq u,\ y \geq v$.

Remark 3.4. For $\varphi(t) = t$ and $\psi(t) = \frac{1-k}{2} \cdot t$ with $0 \leq k < 1$ condition (3.2) reduces to (3.1).

Another generalization of Bhaskar and Lakshmikantham's theorem was given by V. Berinde. In [2] the following class of mappings is introduced.

Remark 3.5. The functional $\varphi : [0,\infty) \to [0,\infty)$ belongs to $\tilde{\Phi}$ if it satisfy the following conditions:

(i_φ) φ is continuous and (strictly) increasing;
(ii_φ) $\varphi(t) < t$ for all $t > 0$;
(iii_φ) $\varphi(t+s) \leq \varphi(t) + \varphi(s),\ \forall t, s \in [0,\infty)$.

As before, we recall that the functional $\psi : [0,\infty) \to [0,\infty)$ belongs to the set Ψ if it satisfy the following conditions:

(i_ψ) $\lim\limits_{t\to r} \psi(t) > 0$ for all $r > 0$;
(ii_ψ) $\lim\limits_{t\to 0_+} \psi(t) = 0$.

Theorem 3.6 (V. Berinde [2])**.** *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *be a mixed monotone mapping for which there exist* $\varphi \in \tilde{\Phi}$ *and* $\psi \in \Psi$ *such that for all* $x, y, u, v \in X$ *with* $x \geq u,\ y \leq v$.

$$\varphi\left(\frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2}\right) \leq$$
$$\leq \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right).$$

Suppose either

(a) *F is continuous mapping, or*
(b) *X satisfies Assumption 1.*

If there exist $x_0, y_0 \in X$ *such that*

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0)$$

or

$$x_0 \geq F(x_0, y_0) \text{ and } y_0 \leq F(y_0, x_0),$$

then there exist $x^*, y^* \in X$ *such that*

$$\begin{cases} x^* = F(x^*, y^*) \\ y^* = F(y^*, x^*) \end{cases}.$$

The first main result of this section is: the following generalization to the multi-valued case of Theorem 3.6.

Theorem 3.7. *Let* $(X, \leq, d)$ *be an ordered b-metric space with constant* $s \geq 1$ *such that the b-metric* d *is complete. Let* $F : X \times X \to P_{cl}(X)$ *be a multi-valued operator having the strict mixed monotone property with respect to "* $\leq$ *". Assume that:*

i) *there exist two functionals* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\varphi\left(H_d(F(x, y), F(u, v)) + H_d(F(y, x), F(v, u))\right) \leq$$
$$\leq \varphi\left(d(x, u) + d(y, v)\right) - \psi\left(d(x, u) + d(y, v)\right)$$

for all $(x, y),\ (u, v) \in X \times X$ *with* $x \leq u,\ y \geq v$.

ii) *there exist* $(x_0, y_0) \in X \times X$ *and* $(x_1, y_1) \in F(x_0, y_0) \times F(y_0, x_0)$ *such that* $x_0 \leq x_1$ *and* $y_0 \geq y_1$.

Then there exist a pair $(x^*, y^*) \in X \times X$ *with*

$$\begin{cases} x^* \in F(x^*, y^*) \\ y^* \in F(y^*, x^*) \end{cases}$$

and two sequences $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *in* X *with*

$$\begin{cases} x_{n+1} \in F(x_n, y_n) \\ y_{n+1} \in F(y_n, x_n) \end{cases}$$

for all $n \in \mathbb{N}$, *such that* $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$.

Proof. Let $Z = X \times X$ and consider on Z the partial order relation " $\leq_p$ " generated by " $\leq$ " and the metric $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by:

$$\tilde{d}((x, y), (u, v)) = d(x, u) + d(y, v),\ \forall (x, y),\ (u, v) \in Z.$$

Then $(Z, \leq_p, \tilde{d})$ is an ordered complete b-metric space.

Consider $G : Z \to P(Z)$, $G(x, y) = F(x, y) \times F(y, x)$, for all $(x, y) \in Z$.

Since F is strict mixed monotone multi-valued operator follow that G is strong increasing multi-valued operator with respect to " $\leq_p$ ".

The multi-valued operator F has a closed value, so the multi-valued operator G has a closed value too.

Let $z = (x, y) \in Z$ and $w = (u, v) \in Z$. We have:

$$H_{\tilde{d}}(G(z), G(w)) = H_{\tilde{d}}\big(F(x, y) \times F(y, x), F(u, v) \times F(v, u)\big) \leq$$
$$\leq H_d\big(F(x, y), F(u, v)\big) + H_d\big(F(y, x), F(v, u)\big)$$

Because φ is increasing we have that

$$\varphi\big(H_{\tilde{d}}(G(z),G(w))\big)\leq\varphi\Big(H_d\big(F(x,y),F(u,v)\big)+H_d\big(F(y,x),F(v,u)\big)\Big)$$
$$\underset{(i)}{\leq}\varphi\left(d(x,u)+d(y,v)\right)-\psi\left(d(x,u)+d(y,v)\right)$$
$$=\varphi(\tilde{d}(z,w))-\psi(\tilde{d}(z,w)).$$

Follow that G is (φ,ψ)-contraction with respect to $\tilde{d}$ for all $z=(x,y)\in Z$ and $w=(u,v)\in Z$ with $z\leq_p w$.

By ii), there exist $z_0=(x_0,y_0)\in Z$ and $z_1=(x_1,y_1)\in F(x_0,y_0)\times F(y_0,x_0)$ such that $x_0\leq x_1$ and $y_0\geq y_1$, so $z_0\leq_{wk}G(z_0)$.

We can apply Theorem 2.2 for the multi-valued operator G and we obtained that $Fix(G)\neq\emptyset$, i.e. there exists $(x^*,y^*)\in Z$ such that $(x^*,y^*)\in G(x^*,y^*)\Rightarrow(x^*,y^*)\in F(x^*,y^*)\times F(x^*,y^*)$ or

$$\begin{cases}x^*\in F(x^*,y^*)\\ y^*\in F(y^*,x^*)\end{cases}$$

and there exists sequence $z_n=(x_n,y_n)\in Z$ of successive approximation of G starting from $(x_0,y_0)\in Z$ such that $(x_n,y_n)\to(x^*,y^*)$ as $n\to\infty$. □

The following result is a global existence and uniqueness theorem for a multi-valued operator.

Theorem 3.8. *Let (X,d) be a complete b-metric space with constant $s\geq 1$. Let $F:X\times X\to P_{cl}(X)$ be a multi-valued operator for which there exist two functionals $\varphi\in\Phi$ and $\psi\in\Psi$ such that*

$$\varphi\left(H_d(F(x,y),F(u,v))+H_d(F(y,x),F(v,u))\right)\leq$$
$$\leq\varphi\left(d(x,u)+d(y,v)\right)-\psi\left(d(x,u)+d(y,v)\right)$$

for all $(x,y),\ (u,v)\in X\times X$. Then the following conclusions hold:

a) *there exist $(x^*,y^*)\in X\times X$ a solution of the coupled fixed point problem (P) and two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X with $x_{n+1}\in F(x_n,y_n)$, $y_{n+1}\in F(y_n,x_n)$ for all $n\in\mathbb{N}$, starting from the arbitrary point $(x_0,y_0)\in X\times X$ and $(x_1,y_1)\in F(x_0,y_0)\times F(y_0,x_0)$ such that $x_n\to x^*$ and $y_n\to y^*$ as $n\to\infty$.*
b) *If, additionally, we suppose that there exists $(u^*,v^*)\in X\times X$ such that $F(u^*,v^*)=\{u^*\}$, $F(v^*,u^*)=\{v^*\}$ and the functional $\psi:[0,\infty)\to[0,\infty)$ is continuous, then $CFix(F)=\{(u^*,v^*)\}$.*

Proof. Consider the functional $\tilde{d}:Z\times Z\to\mathbb{R}_+$, where $Z=X\times X$, defined by

$$\tilde{d}((x,y),(u,v))=d(x,u)+d(y,v),\ \forall(x,y),\ (u,v)\in Z$$

and the operator $G:Z\to P(Z)$ defined by

$$G(x,y)=F(x,y)\times F(y,x),\forall(x,y)\in Z.$$

We can apply Theorem 2.3 of G and we obtain the conclusion a) of this theorem.

b) From the hypotheses $\begin{cases} F(u^*, v^*) = \{u^*\} \\ F(v^*, u^*) = \{v^*\} \end{cases}$ we have $SFix(G) \neq \emptyset$, and because $\psi \in \Psi$ is a continuous mapping, we can apply Theorem 2.3 b) of G and we obtain $SFix(G) = Fix(G) = \{(u^*, v^*)\}$ which means that the coupled fixed problem (P) of F has a unique solution $(u^*, v^*) \in Z$. □

Corollary 3.9. *Let $(X, \leq, d)$ be an ordered b-metric space with constant $s \geq 1$ such that the b-metric d is complete. Let $F : X \times X \to P_{cl}(X)$ be a multi-valued operator having the strict mixed monotone property with respect to "$\leq$". Assume that:*

i) *there exists a functional $\psi \in \Psi$ such that*

$$H_d(F(x, y), F(u, v)) + H_d(F(y, x), F(v, u)) \leq$$
$$\leq d(x, u) + d(y, v) - \psi(d(x, u) + d(y, v))$$

for all $(x, y),\ (u, v) \in X \times X$ with $x \leq u,\ y \geq v$.

ii) *there exist $(x_0, y_0) \in X \times X$ and $(x_1, y_1) \in F(x_0, y_0) \times F(y_0, x_0)$ such that $x_0 \leq x_1$ and $y_0 \geq y_1$.*

Then, there exist a pair $(x^*, y^*) \in X \times X$ with

$$\begin{cases} x^* \in F(x^*, y^*) \\ y^* \in F(y^*, x^*) \end{cases}$$

and two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X with

$$\begin{cases} x_{n+1} \in F(x_n, y_n) \\ y_{n+1} \in F(y_n, x_n) \end{cases}$$

for all $n \in \mathbb{N}$, such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

Proof. In Theorem 3.7 we take $\varphi(t) = t,\ \forall t \in [0, \infty)$ and hence we get Corollary 3.9. □

4. Properties of the solutions of the coupled fixed point problem

Theorem 4.1 (Data dependence)**.** *Let (X, d) be a complete b-metric space with constant $s \geq 1$. Let $F : X \times X \to P_{cl}(X)$ and $S : X \times X \to P_{cl}(X)$ be two multi-valued operators. Suppose that:*

i) *there exist two functionals $\varphi \in \Phi$ and $\psi \in \Psi$, ψ is continuous such that*

$$\varphi(H_d(F(x, y), F(u, v)) + H_d(F(y, x), F(v, u))) \leq$$
$$\leq \varphi(d(x, u) + d(y, v)) - \psi(d(x, u) + d(y, v))$$

for all $(x, y),\ (u, v) \in X \times X$.

ii) *there exists $(x^*, y^*) \in X \times X$ such that*

$$\begin{cases} F(x^*, y^*) = \{x^*\} \\ F(y^*, x^*) = \{y^*\} \end{cases}$$

iii) *there exists $(u^*, v^*) \in X \times X$ such that*

$$\begin{cases} u^* \in S(u^*, v^*) \\ v^* \in S(v^*, u^*) \end{cases}$$

iv) *there exists* $\eta > 0$ *such that* $H_d(F(x,y), S(x,y)) \leq \eta,\ \forall (x,y) \in X \times X$.

v)

$$\begin{cases} \varphi(t) - s\varphi(t) + s\psi(t) > 0,\ \forall t > 0 \\ \varphi(t) - s\varphi(t) + s\psi(t) = 0 \Rightarrow t = 0 \end{cases}$$

Then, we have the following estimation:

$$\varphi_{\tilde{d}}(CFix(S), CFix(F)) \leq \sup\{t \in \mathbb{R}_+ / \varphi(t) - s\varphi(t) + s\psi(t) \leq 2s\eta\}$$

Proof. Let $(u^*, v^*) \in X \times X$ such that

$$\begin{cases} u^* \in S(u^*, v^*) \\ v^* \in S(v^*, u^*) \end{cases}$$

and $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} F(x^*, y^*) = \{x^*\} \\ F(y^*, x^*) = \{y^*\} \end{cases}$$

We denote by $Z = X \times X$ and consider the following functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}\,((x,y),(u,v)) = d(x,u) + d(y,v),\ \forall (x,y),(u,v) \in Z.$$

We have:

$$\begin{aligned} \tilde{d}((x^*,y^*),(u^*,v^*)) &= D_{\tilde{d}}((u^*,v^*), F(x^*,y^*) \times F(y^*,x^*)) \\ &= D_d(u^*, F(x^*,y^*)) + D_d(v^*, F(y^*,x^*)) \\ &\leq H_d(S(u^*,v^*), F(x^*,y^*)) + H_d(S(v^*,u^*), F(y^*,x^*)) \\ &\leq s\,[H_d(S(u^*,v^*), F(u^*,v^*)) + H_d(F(u^*,v^*), F(x^*,y^*))] \\ &\quad + s\,[H_d(S(v^*,u^*), F(v^*,u^*)) + H_d(F(v^*,u^*), F(y^*,x^*))] \\ &\leq 2s\eta + s\,[H_d(F(x^*,y^*), F(u^*,v^*)) + H_d(F(y^*,x^*), F(v^*,u^*))] \end{aligned}$$

On the other hand, φ is increasing, so:

$$\begin{aligned} &\varphi(\tilde{d}((x^*,y^*),(u^*,v^*))) \\ &\leq 2s\eta + s\varphi(H_d(F(x^*,y^*), F(u^*,v^*)) + H_d(F(y^*,x^*), F(v^*,u^*))) \\ &\leq 2s\eta + s\varphi(\tilde{d}((x^*,y^*),(u^*,v^*))) - s\psi(\tilde{d}((x^*,y^*),(u^*,v^*))) \end{aligned}$$

$$\Rightarrow \varphi(\tilde{d}((x^*,y^*),(u^*,v^*))) - s\varphi(\tilde{d}((x^*,y^*),(u^*,v^*))) + s\psi(\tilde{d}((x^*,y^*),(u^*,v^*))) \leq 2s\eta.$$

That means

$$\tilde{d}((x^*,y^*),(u^*,v^*)) \leq \sup\{t \in \mathbb{R}_+ / \varphi(t) - s\varphi(t) + s\psi(t) \leq 2s\eta\}$$

$$\Rightarrow D_{\tilde{d}}((u^*,v^*), CFix(F)) \leq \sup\{t \in \mathbb{R}_+ / \varphi(t) - s\varphi(t) + s\psi(t) \leq 2s\eta\}$$

$$\Rightarrow \varphi_{\tilde{d}}(CFix(S), CFix(F)) \leq \sup\{t \in \mathbb{R}_+ / \varphi(t) - s\varphi(t) + s\psi(t) \leq 2s\eta\} \qquad \square$$

Consider the system of inclusions

$$\begin{cases} x \in F(x,y) \\ y \in F(y,x) \end{cases} \tag{4.1}$$

and $\tilde{d} : Z \times Z \to \mathbb{R}_+$, where $Z = X \times X$, defined by:

$$\tilde{d}((x,y),(u,v)) = d(x,u) + d(y,v),\ \forall (x,y),\ (u,v) \in Z.$$

By definition, the system of inclusions (4.1) is well-posed with respect to $D_{\tilde{d}}$ if:

(i) there exists $w^* = (u^*, v^*) \in Z$ such that

$$\begin{cases} F(u^*, v^*) = \{u^*\} \\ F(v^*, u^*) = \{v^*\} \end{cases}$$

(ii) if $w_n = (u_n, v_n)_{n\in\mathbb{N}}$ is a sequence in Z with the following properties $D_d(u_n, F(u_n, v_n)) \to 0$ respectively $D_d(v_n, F(v_n, u_n)) \to 0$ as $n \to \infty$, then $d(u_n, u^*) + d(v_n, v^*) \to 0$ as $n \to \infty$ or $\tilde{d}((u_n, v_n), (u^*, v^*)) \to 0$ as $n \to \infty$.

Theorem 4.2 (Well-posedness)**.** *We suppose that all the hypotheses of Theorem 3.8 take place. If additionally we have:*

$$\begin{cases} \varphi(t) - s\varphi(t) + s\psi(t) > 0, \forall t > 0 \ (*) \\ \varphi(t_n) - s\varphi(t_n) + s\psi(t_n) \to 0 \Rightarrow t_n \to 0 \ \textit{as}\ n \to \infty \ (**) \end{cases}$$

Then, the system of inclusions (4.1) is well-posed with respect to D_d.

Proof. From Theorem 3.8 follows that the system of inclusions (4.1) has a unique solution $w^* = (u^*, v^*) \in Z$.

Let $w_n = (u_n, v_n)_{n\in\mathbb{N}}$ in Z with $D_d(u_n, F(u_n, v_n)) \to 0$, respectively $D_d(v_n, F(v_n, u_n)) \to 0$ as $n \to \infty$.

We have:

$$\begin{aligned} \tilde{d}((u_n, v_n), (u^*, v^*)) &= D_{\tilde{d}}((u_n, v_n), F(u^*, v^*) \times F(v^*, u^*)) \\ &= D_d(u_n, F(u^*, v^*)) + D_d(v_n, F(v^*, u^*)) \\ &\le s\left[D_d(u_n, F(u_n, v_n)) + H_d(F(u_n, v_n), F(u^*, v^*))\right] \\ &\quad + s\left[D_d(v_n, F(v_n, u_n)) + H_d(F(v_n, u_n), F(v^*, u^*))\right]. \end{aligned}$$

From φ is an increasing mapping follows that:

$$\begin{aligned} \varphi(\tilde{d}((u_n, v_n), (u^*, v^*))) &\le sD_{\tilde{d}}((u_n, v_n), F(u_n, v_n) \times F(v_n, u_n)) + \\ &\quad + s\varphi(H_d(F(u_n, v_n), F(u^*, v^*)) + H_d(F(v_n, u_n), F(v^*, u^*))) \\ &\le sD_{\tilde{d}}((u_n, v_n), F(u_n, v_n) \times F(v_n, u_n)) + \\ &\quad + s\varphi(\tilde{d}((u_n, v_n), (u^*, v^*))) - s\psi(\tilde{d}((u_n, v_n), (u^*, v^*))) \end{aligned}$$

We get

$$\begin{aligned} &\varphi(\tilde{d}((u_n, v_n), (u^*, v^*))) - s\varphi(\tilde{d}((u_n, v_n), (u^*, v^*))) + s\psi(\tilde{d}((u_n, v_n), (u^*, v^*))) \le \\ &\le sD_{\tilde{d}}((u_n, v_n), F(u_n, v_n) \times F(v_n, u_n)). \end{aligned}$$

Letting $n \to \infty$ we get

$$\lim_{n\to\infty}[\varphi(\tilde{d}((u_n,v_n),(u^*,v^*)))-s\varphi(\tilde{d}((u_n,v_n),(u^*,v^*)))+s\psi(\tilde{d}((u_n,v_n),(u^*,v^*)))]=0$$

By the assumptions (*) and (**) we get that $\tilde{d}((u_n,v_n),(u^*,v^*)) \to 0$ as $n \to \infty$. □

In what follows we give an Ulam-Hyers stability property.

Definition 4.3. Let (X,d) be a b-metric space with constant $s \geq 1$ and $F : X \times X \to P(X)$ be a multi-valued operator. Let $\tilde{\mathrm{d}}$ be any b-metric on $Z = X \times X$ generated by d. By definition, the system of inclusions (4.1) is Ulam-Hayers stable if there exists an increasing operator $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0 with $\gamma(0) = 0$ such that for each $\varepsilon \in \mathbb{R}_+^*$ and for each solution $(\bar{x},\bar{y}) \in Z$ of the inequality $D_{\tilde{d}}((x,y),F(x,y) \times F(y,x)) \leq \varepsilon$ there exists a solution $(x^*,y^*) \in Z$ of the system of inclusions (4.1) such that $\tilde{d}((x^*,y^*),(\bar{x},\bar{y})) \leq \gamma(\varepsilon)$.

Theorem 4.4 (Ulam-Hyers stability)**.** *Consider the system of inclusions (4.1). Let us suppose that all the hypotheses of Theorem 3.8 take place. If additionally we have:*
$\varphi(t) - s\varphi(t) + s\psi(t) > 0,\ \forall t > 0$
$\varphi(t) - s\varphi(t) + s\psi(t) = 0 \Rightarrow t = 0,$
then, the system of inclusions (4.1) is Ulam-Hyers stable.

Proof. Using Theorem 3.8 we obtain that there exists a unique pair $(x^*,y^*) \in Z$ such that
$\{x^*\} = F(x^*,y^*)$
$\{y^*\} = F(y^*,x^*)$.
Consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}\,((x,y),(u,v)) = d(x,u) + d(y,v),\ \forall (x,y),(u,v) \in Z.$$

Then we get

$$\begin{aligned}
\tilde{d}((\bar{x},\bar{y}),(x^*,y^*)) &= d(\bar{x},x^*) + d(\bar{y},y^*) \\
&= D_d(\bar{x},F(x^*,y^*)) + D_d(\bar{y},F(y^*,x^*)) \\
&\leq s\,[D_d(\bar{x},F(\bar{x},\bar{y})) + H_d(F(\bar{x},\bar{y}),F(x^*,y^*))] \\
&\quad + s\,[D_d(\bar{y},F(\bar{y},\bar{x})) + H_d(F(\bar{y},\bar{x}),F(y^*,x^*))] \\
&\leq s\varepsilon + s\,[H_d(F(\bar{x},\bar{y}),F(x^*,y^*)) + H_d(F(\bar{y},\bar{x}),F(y^*,x^*))]
\end{aligned}$$

Because φ is an increasing mapping we get that

$$\varphi(\tilde{d}((\bar{x},\bar{y}),(x^*,y^*))) \leq s\varphi(\tilde{d}((\bar{x},\bar{y}),(x^*,y^*))) - s\psi(\tilde{d}((\bar{x},\bar{y}),(x^*,y^*))) + s\varepsilon,$$

and so

$$\tilde{d}((\bar{x},\bar{y}),(x^*,y^*)) \leq \sup\{t \in \mathbb{R}_+/\varphi(t) - s\varphi(t) + s\psi(t) \leq s\varepsilon\} := \gamma(\varepsilon).$$ □

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Manuscript received 14 November 2016
revised 16 January 2017

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