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NONLINEAR SCALARIZING FUNCTIONALS FOR COMPUTING MINIMAL POINTS UNDER VARIABLE ORDERING STRUCTURES

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ABSTRACT. We consider the problem of finding minimal points to a subset of a Banach space with respect to a variable domination structure given by a set-valued mapping. Our aim is to present a new scalarization method that characterizes these minimal points as well as the related weak concept with respect to a variable domination structure. Properties of the scalarizing functional such as monotonicity, convexity and semi-continuity are studied. The application of the scalarization method for computing weakly minimal points with respect to a variable domination structure is also discussed.

1. INTRODUCTION

Consider a subset A of a Banach space Y. The problem of determining minimal elements of A with respect to a certain comparison criterion is to find $a^* \in A$ such that there is no $a \in A \setminus \{a^*\}$ satisfying that a is *better* than a^*. In classical vector optimization we consider a set A of images of an objective function $F : X \to Y$ over a feasible set $C \subseteq X$ and an element $F(x_1)$ is better than $F(x_2)$ if $F(x_2) - F(x_1) \in K \setminus \{0\}$, where K is a proper, closed, convex and pointed cone and $x_1, x_2 \in C$. The concept of Pareto minimality says that for $x^* \in C$ there is no $x \in C$ such that $F(x^*) - F(x) \in K \setminus \{0\}$. Furthermore, an element $F(x^*)$ is called a weakly minimal element of A if $F(x^*) - F(x)$ cannot be an element of the interior of K for all $x \in C$. Weakly minimal and minimal elements of A are characterized by means of scalar functionals. An important class of scalarizing functionals $\psi : Y \times [Y \setminus \{0\}] \to \mathbb{R} \cup \{-\infty, +\infty\}$ is given by

$$\psi(y, r) = \inf\{t \in \mathbb{R} \ : \ y + tr \in A + K\},$$

where $r \in Y \setminus \{0\}$. This functional is intensively studied in the literature (see [18–20] and references therein).

More general, it is possible to use variable domination structures in vector optimization. Given a set-valued mapping $K : Y \rightrightarrows Y$, such that $K(y)$ is a closed, convex and pointed (i.e., $K(y) \cap (-K(y)) = \{0\}$) set with $K(y) \neq \{0\}$ for all $y \in Y$, the problem of computing minimal points of A with respect to the domination structure induced by K is to find $a \in A$, such that

$$A \cap (a - K(a)) = \{a\}. \tag{1.1}$$

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This model generalizes the classical vector optimization that corresponds to the case in which $K(y) \equiv K$ for all $y \in Y$, where $K \subset Y$ is a proper, convex, closed and pointed cone.

Recently, vector (as well as set) optimization problems with variable ordering structure are studied intensively in the literature, see the book [16] and references therein, [3, 7–11, 13–15, 24, 25] and [26]. Important applications of vector optimization with variable domination structure appear in medical image registration, psychological modeling, economics, portfolio optimization and location theory, see for instance [1, 4, 12, 17, 30].

An interesting application of vector optimization with variable domination structures is given in the theory of consumer demand in economics by John [21, 22]. In John [21, 22] and references therein (compare Eichfelder [16]) a local and a global theory is presented in order to explain consumer behavior. In the local approach, it is supposed that the consumer faces a nonempty set of feasible alternatives $A \subset \mathbb{R}^m$ from which he is allowed to choose. By contrast, with the global approach, a local preference only requires that the consumer is able to rank alternatives in a small neighborhood of a given commodity bundle relative to that bundle. This idea can be represented by an economical comparative function $g : \mathbb{R}^m \to \mathbb{R}^m$ such that y belonging to a neighborhood of $\overline{y}$ is interpreted to be better than $\overline{y}$ if and only if $g(\overline{y})(y - \overline{y}) < 0$. For $A \subset \mathbb{R}^m$, the choice set assigned to A in the local theory is then characterized by (see [22])

$$C(A) := \{\overline{y} \in A \ : \ \forall y \in A : g(\overline{y})(y - \overline{y}) \geq 0\}.$$

This leads us to a set-valued map $K : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined by (for $\overline{y} \in A$):

$$K(\overline{y}) := \{d \in \mathbb{R}^m \ : \ g(\overline{y})(d) \geq 0\}.$$

If the consumer is interested in an alternative $\overline{y} \in A$ such that

$$\forall y \in A : \quad g(\overline{y})(y - \overline{y}) \geq 0,$$

then $y - \overline{y} \in K(\overline{y})$ for all $y \in A$, i.e., $A \subset \{\overline{y}\} + K(\overline{y})$.

Furthermore, if the consumer is looking for alternatives $\overline{y} \in A$ such that

$$\forall y \in A \setminus \{\overline{y}\} : \quad g(\overline{y})(\overline{y} - y) < 0,$$

then, for all $y \in A \setminus \{\overline{y}\}$,

$$\overline{y} - y \notin K(\overline{y}).$$

This means that the consumer is looking for alternatives $\overline{y} \in A$ such that

$$A \cap (\overline{y} - K(\overline{y})) = \{\overline{y}\},$$

i.e., the consumer is looking for minimal points of a vector optimization problem with variable domination structure in the sense of (1.1).

In this paper we consider the problem of finding minimal points of a set A with respect to the variable domination structure given by the mapping K. Our aim is to present a scalarization method that characterizes these minimal points as well as the related weak concept with respect to a variable domination structure. Properties of the scalarizing functional such as monotonicity, convexity and semi-continuity are studied. The application of the scalarizing functional to the practical computation of weakly minimal points of a set A with respect to a variable domination structure given by the mapping K is also discussed.

The paper is organized as follows. First, in Section 2, the solution concepts for vector optimization problems with variable domination structure are introduced. Section 3 is devoted to the presentation of the main results, especially we derive continuity properties of the scalarizing functional. A descent algorithm for computing minimal points of a set A is presented in Section 4. The paper ends with a solution approach for the case in which A is the image set of a differentiable function.

We end this section with some notations and concepts which will be useful for the study of the scalarization functional.

From now on X and Y are Banach spaces, $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ and, as usual, $-\infty + a = -\infty$ for all $a \in \mathbb{R}$ and $a(-\infty) = -\infty$, if $a > 0$. We denote as $[a, z]$ the set $\{u = \alpha a + (1-\alpha)z \ : \ \alpha \in [0,1]\}$, $B(a, \varepsilon)$ is the open ball centered in a with radius ε and $V(y)$ represents a neighborhood of y. Analogously if $A \subset Y$, $V(A)$ is a neighborhood of A. If $y^* \in Y$, and $A \subset Y$, $y^* \pm A := \{y \in Y : \ y = y^* \pm A\}$. If Y is a Hilbert space, $\langle \cdot, \cdot \rangle$ represents the corresponding scalar product. Finally, if Y is a finite dimensional space, we assume $\| \cdot \|$ is the Euclidean norm.

Let $\Phi : X \rightrightarrows Y$ be a set-valued mapping. If $D \subset X$, the image of D by the mapping Φ is denoted by $\Phi(D)$ ($\Phi(D) := \cup_{x \in D} \Phi(x)$). We will use the following definitions of semi-continuity and closedness, see [2] for more details.

Definition 1.1. Let $\Phi : X \rightrightarrows Y$ be a set-valued map, where X and Y are Banach spaces.

Φ is **Berge-upper semi-continuous (B-usc) at $\overline{\mathbf{x}} \in \mathbf{X}$** if for each open subset V of Y such that $\Phi(\overline{x}) \subset V$, there is a neighborhood U of $\overline{x}$ satisfying $\Phi(U) \subset V$.

Φ is **Berge-lower semi-continuous (B-lsc) at $\overline{\mathbf{x}} \in \mathbf{X}$** if for each open subset V of Y such that $\Phi(\overline{x}) \cap V \neq \emptyset$ there is a neighborhood U of $\overline{x}$ satisfying $\Phi(U) \cap V \neq \emptyset$ for all $x \in U$.

Φ is **closed at $\overline{\mathbf{x}} \in \mathbf{X}$** if for all $x_n \to \overline{x}$, $y_n \to \overline{y}$, $y_n \in \Phi(x_n)$ implies that $\overline{y} \in \Phi(\overline{x})$. Φ is **closed** if it is closed at every $x \in X$, i.e., if $\mathrm{gph}(\Phi) = \{(x, y) \in X \times Y \ : \ y \in \Phi(x)\}$ is a closed set.

More details on these concepts, as well as their relationships with other semicontinuity definitions, can be found in [2].

In the following section, we introduce the model we will deal with and a useful scalarization technique.

2. Solution concepts and scalarization methods

In this section, we will study the problem of finding the minimal points to a subset A of Y with respect to a variable domination structure. Definitions and some properties will be given. Unless something else is stated, we assume that

- Y is a Banach space and A is a nonempty subset of Y.

- The variable domination structure is given by the set-valued mapping $K : Y \rightrightarrows Y$, where $K(y)$ is a convex, closed, pointed set, $K(y) \neq \{0\}$ for all $y \in Y$.

Recall that a set $K \subset Y$ is pointed if $K \cap -K = \{0\}$.

An important type of domination structure appears when the mapping $K : Y \rightrightarrows Y$ is such that $K(y)$ is a cone for all $y \in Y$, i.e., K is a cone-valued mapping. If $K(y)$ is a proper, closed, convex and pointed cone for all $y \in Y$, the relation $a \succeq_y b$ if and only if $a - b \in K(y)$, defines a partial order, which is also stable with respect to the sum and the positive scalar multiplication in Y. For this reason, we will particularize the more generally obtained results to the cone valued ordering case.

Let a domination structure be given by a mapping K. This means that points on Y are compared in the sense that $y^1 \succeq y^2$ if and only if $y^1 - y^2 \in K(y^1)$. Using the domination mapping $K : Y \rightrightarrows Y$, for a given set $A \subset Y$, we can study the points $\overline{y} \in A$ such that there exists no point $y(\neq \overline{y}) \in A$, fulfilling $\overline{y} - y \in K(\overline{y})$. This leads to the following concept of minimal elements of a set A (compare Eichfelder [16]):

Definition 2.1. Assume that $A \subset Y$ and $K : Y \rightrightarrows Y$ is a set-valued mapping such that $K(y)$ is a closed, convex, pointed set, $K(y) \neq \{0\}$ for each $y \in Y$. An element $\overline{y} \in A$ is a **minimal point of A with respect to K**, if

$$A \cap (\overline{y} - K(\overline{y})) = \{\overline{y}\}.$$

The **set of minimal points of A with respect to K** is denoted by $\mathrm{Min}(A, K(\cdot))$.

Corresponding to this definition, we will denote the problem of finding minimal points of a set A with respect to a variable domination structure given by $K : Y \rightrightarrows Y$ as

$$\mathrm{Min}(A, K(\cdot)). \tag{2.1}$$

This model will be called geometric vector optimization problem with variable domination structure. The elements of $\mathrm{Min}(A, K(\cdot))$ are the solutions of (2.1).

As in the case in which $K(\cdot) \equiv K$ (that is $K(\cdot)$ is constant), weakly minimal points of A with respect to K are very important from the theoretical as well as practical viewpoint. We will introduce them in the following definition.

Definition 2.2. Assume that $A \subset Y$ and $K : Y \rightrightarrows Y$ is a set-valued mapping such that $K(y)$ is a closed, convex, pointed set and $\mathrm{int}(K(y)) \neq \emptyset$ for each $y \in Y$. An element $\overline{y} \in A$ is called **weakly minimal point of A with respect to K** if

$$A \cap (\overline{y} - \mathrm{int}(K(\overline{y}))) = \emptyset.$$

The **set of weakly minimal points of A with respect to K** is denoted by $\mathrm{WMin}(A, K(\cdot))$.

For more details about solution concepts for vector optimization problems with variable domination structures, see Eichfelder [16].

Let us illustrate these definitions in an example:

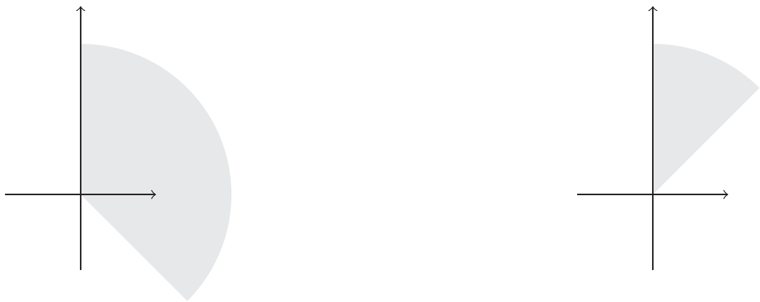

FIGURE 1. The cones K_1 and K_2 defining the domination map K in Example 2.3.

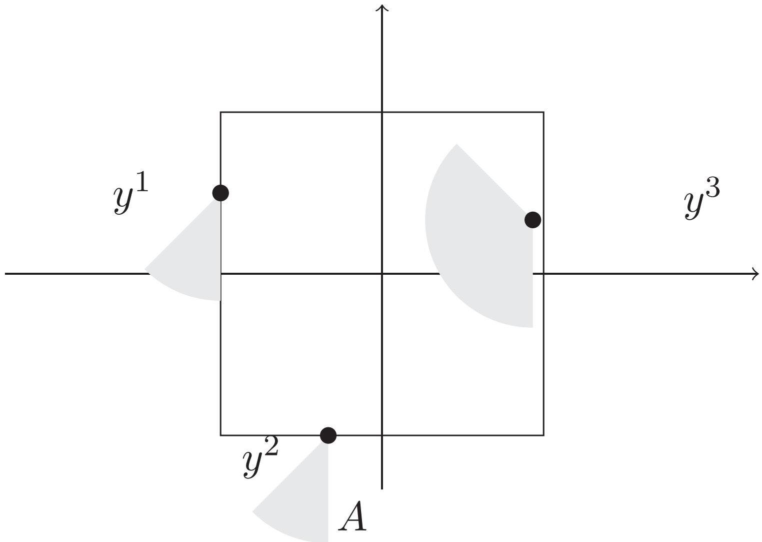

FIGURE 2. Elements y^1, y^2, y^3 of A and the corresponding ordering cones $K(y^i)$, $i = 1, 2, 3$, in Example 2.3.

Example 2.3. Consider $A = [-1, 1] \times [-1, 1]$ and the domination map

$$K(y) := \begin{cases} K_1 & \text{if } y_1 \geq 0, \\ K_2 & \text{otherwise,} \end{cases}$$

where

$$K_1 := \{z \in \mathbb{R}^2 : z_1 \geq 0, z_2 \geq -z_1\},$$

$$K_2 := \mathbb{R}^2_+ \cap \{z \in \mathbb{R}^2 : z_1 - z_2 \leq 0\}.$$

Figure 1 shows the cones K_1 and K_2 which define the domination map K.

Furthermore, Figure 2 depicts the sets A and $y - K(y)$ for the points y^1, y^2 and y^3. Evidently, $y^1 \in \mathrm{WMin}(A, K(\cdot)) \setminus \mathrm{Min}(A, K(\cdot))$, $y^2 \in \mathrm{Min}(A, K(\cdot))$ and $y^3 \notin \mathrm{WMin}(A, K(\cdot))$.

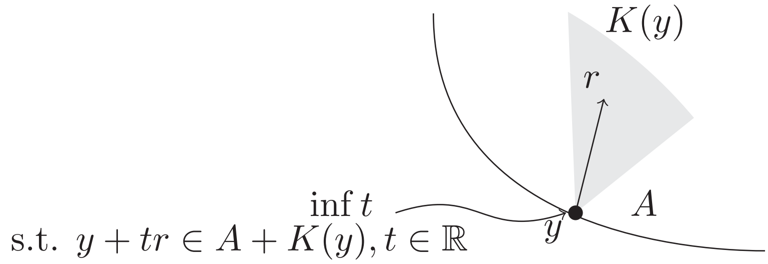

FIGURE 3. The scalarizing functional $\psi(y, r)$ given by (3.1) for $y = 0$.

In fact the set of minimal points is

$$\mathrm{Min}(A, K(\cdot)) = \{(y_1, -1) \in \mathbb{R}^2 : \ y_1 \in [-1, 0)\},$$

and the set of weakly minimal elements is

$$\mathrm{WMin}(A, K(\cdot)) = \{(-1, y_2) \in \mathbb{R}^2 : y_1 \in [-1, 0), y_2 \in [-1, 1]\} \cup \{(y_1, -1) \in \mathbb{R}^2\}.$$

It is clear after the definition that $\mathrm{Min}(A, K(\cdot)) \subseteq \mathrm{WMin}(A, K(\cdot))$. As in the case of a classical vector optimization problem, there exist weakly minimal points of A such as $(-1, 1)$, which are not close to the set of $\mathrm{Min}(A, K(\cdot))$.

As in the classical case of vector optimization, an appealing approach is to use scalar functionals for characterizing the elements of $\mathrm{Min}(A, K(\cdot))$ and $\mathrm{WMin}(A, K(\cdot))$. A scalarization for model (2.1) is introduced in the next section.

3. A nonlinear scalarization method for vector optimization problems with variable domination structure

In this section, the problem of finding (weakly) minimal points of the set A with respect to the set-valued map $K : Y \rightrightarrows Y$ is considered. As already pointed out, scalarizations are widely used in vector optimization models, see [20]. Also for the case of variable domination structure, scalarization methods are studied in the literature (see [9–11, 16, 24] and references therein). Our paper is devoted to the study of a new nonlinear scalarization by means of a functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ defined by

$$\psi(y, r) := \inf\{t \in \mathbb{R} \ : \ y + tr \in A + K(y)\}, \tag{3.1}$$

where $y \in Y, r \in Y \setminus \{0\}$.

The example in Figure 3 shows the value of $\psi(y, r)$ for $y \in A$ and $r \in K(y)$. Furthermore, it holds that $\psi(y, r) = 0$ for all $r \in K(y)$ at a minimal point $y \in A$, which is a desirable property of this class of functionals.

In the first subsection, a characterization of minimal and weakly-minimal points by means of the scalarizing functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ will be derived. Roughly speaking, we will present sufficient conditions for the property $\psi(\overline{y}, r) = 0$ for $\overline{y} \in \mathrm{Min}(A, K(\cdot))$ and for all $r \in K(\overline{y}) \setminus \{0\}$. Furthermore, we will formulate assumptions under which we can guarantee that the elements of the form $\overline{y} = y + \psi(y, r)r - k \in A$ are weakly minimal points of A with respect to K. The second subsection includes the analysis of algebraic and topological properties of the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$. Sufficient conditions for the semi-continuity and the convexity of ψ are presented.

3.1. **Characterization of (weakly) minimal points.** In this part, we will study the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ given by (3.1) and its relationship to weakly minimal and minimal points of $A \subset Y$ with respect to $K : Y \rightrightarrows Y$. We use the convention that $\psi(y, r) = +\infty$ if $\{y + tr \ : \ t \in \mathbb{R}\} \cap (A + K(y)) = \emptyset$. As already supposed, Y is a Banach space.

Before analyzing the finiteness of ψ, we introduce a related concept of boundedness of a set A.

Definition 3.1. Let Y be a Banach space. $A \subset Y$ is said to be **bounded** if there exists $M > 0$ such that $A \subset B(0, M)$.

Now, we present the result concerning the finiteness of the functional ψ.

Proposition 3.2. *Let $A \subset Y$ and $K : Y \rightrightarrows Y$ be a set-valued map such that $K(y)$ is a convex, closed and pointed set, $K(y) \neq \{0\}$ for all $y \in Y$. We consider the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$, defined in* (3.1). *Then*

(i) $\psi(y, r) \leq 0$, *for all $y \in A$, $r \in K(y) \setminus \{0\}$.*

(ii) *Suppose that either*

(A1) *$A + K(y)$ is a bounded set,*

or

(A2) $\dim(Y) < +\infty$, *A bounded.*

Then for all fixed elements $y \in A$ and for all $r \in K(y) \setminus \{0\}$, it holds that $\psi(y, r) \in \mathbb{R}$.

Proof. (i) Since $K(y)$ is pointed, it holds that $0 \in K(y)$. So, due to $y \in A$, we have that $y \in A + K(y)$. Therefore, $\psi(y, r) \leq 0 < +\infty$ for all $r \in K(y) \setminus \{0\}$, $y \in A$.

(ii) By (i), we only need to prove that $\psi(y, r) > -\infty$ for all fixed $y \in A$ and $r \in K(y) \setminus \{0\}$. Define the set

$$T(y, r) := \{t \in \mathbb{R} : \ y + tr \in A + K(y)\}.$$

We consider two cases.

Case 1: If (A1) holds, it is clear that the set $A + K(y) - y$ is bounded and, as $r \neq 0$, $T(y, r)$ must be bounded. So, $\psi(y, r) \in \mathbb{R}$ is finite for all fixed $y \in A$ and all $r \in K(y) \setminus \{0\}$.

Case 2: Suppose that (A2) holds. Then we have that A is bounded and $\dim(Y) < +\infty$. So we can take $\| \cdot \|$ as the Euclidean norm. We consider a fixed element $y \in A$ and an arbitrary element $r \in K(y) \setminus \{0\}$. Then, since A is bounded, the set $\{z \in A : z \in y + tr - K(y) \text{ for some } t \in \mathbb{R}\}$, is bounded.

Suppose that there are $y \in A$, fixed, and $r \in K(y) \setminus \{0\}$ with $\psi(y, r) = -\infty$. Then there exist two sequences $\{t_n\}$, $t_n \to -\infty$, and $\{k_n\}$, $k_n \in K(y) \setminus \{0\}$, such that for $n = 1, 2, \ldots$

$$y + t_n r - k_n \in A.$$

Since A is bounded, $\{t_n r - k_n\}$ is a bounded sequence. This implies,

$$\|k_n\| \to +\infty, \tag{3.2}$$

since $t_n \to -\infty$. W.l.o.g. we assume that $\|r\| = 1$. Now, we define $u_n := \frac{k_n}{\|k_n\|} \in Y$, i.e., $k_n = \alpha_n u_n$ where $\alpha_n := \|k_n\| > 0$. Hence, $\|u_n\| = \|r\| = 1$ and, by (3.2), $\alpha_n \to +\infty$.

Note that, by the pointedness of $K(y)$, we have $0 \in K(y)$. Since $\alpha_n u_n = k_n \in K(y)$ and for n large enough, $\alpha_n > 1$, u_n can be written as a convex combination of these two elements of $K(y)$ as follows:

$$u_n = \frac{1}{\alpha_n}(\alpha_n u_n) + \left(1 - \frac{1}{\alpha_n}\right) 0.$$

Taking into account that $K(y)$ is a convex set, it holds that $u_n \in K(y)$.
Moreover, due to the closedness of $K(y)$, for each limit point u of the sequence $\{u_n\}$, it holds that $u \in K(y)$. Now, we consider an upper bound $M > 0$ of $\{\|-t_n r + k_n\|^2\}$. Then it holds that

$$\begin{aligned} M \geq \| - t_n r + k_n\|^2 &= t_n^2 - 2t_n\alpha_n\langle r, u_n\rangle + \alpha_n^2 \\ &= (t_n + \alpha_n)^2 - 2t_n\alpha_n(1 + \langle r, u_n\rangle) \\ &\geq -2t_n\alpha_n(1 + \langle r, u_n\rangle). \end{aligned}$$

Hence

$$M \geq -2t_n\alpha_n(1 + \langle r, u_n\rangle). \tag{3.3}$$

This means that $\{-2t_n\alpha_n(1 + \langle r, u_n\rangle)\}$ is upper bounded. Since $t_n \to -\infty$ and $\alpha_n \to +\infty$, it holds that $(-2t_n)\alpha_n \to +\infty$. Since $\|u_n\| = \|r\| = 1$, by the Cauchy-Schwartz inequality, $(1 + \langle r, u_n\rangle) \geq 0$. So, by (3.3)

$$\lim_{n\to+\infty}(1 + \langle r, u_n\rangle) = 0.$$

This means that $\langle r, u\rangle = -1$. Since $\|r\| = \|u\| = 1$, we get $u = -r \in -K(y) \setminus \{0\}$, contradicting the pointedness of $K(y)$. So, there exists no sequence $\{t_n\}$, $t_n \in T(y, r)$, such that $t_n \to -\infty$ and hence $\psi(y, r) > -\infty$ for all fixed $y \in A$ and for all $r \in K(y) \setminus \{0\}$. $\square$

Remark 3.3. If K is a cone-valued map, (A2) guarantees the finiteness of the functional ψ at $(y, r) \in A \times [K(y) \setminus \{0\}]$.

Now, we will use the functional ψ, given by (3.1), to present a condition which must be satisfied by minimal elements of A with respect to the variable domination map $K(\cdot)$.

Lemma 3.4. *Given $\overline{y} \in Y$ and the domination map $K : Y \rightrightarrows Y$, where $K(y)$ is a convex, closed and pointed set, $K(y) \neq \{0\}$ for all $y \in Y$, suppose that either $K(\overline{y})$ is a cone, or A is a convex set.*
If $\overline{y} \in \mathrm{Min}(A, K(\cdot))$, then it holds that $\psi(\overline{y}, r) = 0$ for all $r \in K(\overline{y}) \setminus \{0\}$.

Proof. Consider $\overline{y} \in \mathrm{Min}(A, K(\cdot))$, $r \in K(\overline{y}) \setminus \{0\}$. By Proposition 3.2(i), we have that $\psi(\overline{y}, r) \leq 0$ for $\overline{y} \in A$ and for all $r \in K(\overline{y}) \setminus \{0\}$, since $K(\overline{y})$ is pointed and $\overline{y} \in A$. Suppose that $\psi(\overline{y}, r) < 0$, then there exist $y \in A$, $y \neq \overline{y}$ and $t < 0$ such that

$$\overline{y} + tr = y + k, \text{ for some } k \in K(\overline{y}). \tag{3.4}$$

Now, we consider two cases.

Case 1: If $K(\overline{y})$ is a cone, as $t<0$, we have that $-tr \in K(\overline{y}) \setminus \{0\}$. But $K(\overline{y})$ is also convex and pointed, so $-tr+k \in K(\overline{y}) \setminus \{0\}$ for $k \in K(\overline{y})$ and

$$\overline{y} \in y + K(\overline{y}) \setminus \{0\},$$

contradicting the assumption that $\overline{y} \in \mathrm{Min}(A, K(\cdot))$.

Case 2: Here, we assume that A is convex. First, we take $r, k, 0 \in K(\overline{y})$. By the convexity of $K(\overline{y})$, it is possible to chose $\alpha \in (0,1)$ (depending from $t<0$) such that $\alpha(-tr+k) \in K(\overline{y})$. But, as A is convex and $y \in A$, we get that

$$[\alpha y + (1-\alpha)\overline{y}] \in A.$$

Note that $\alpha \neq 0$ and so $[\alpha y + (1-\alpha)\overline{y}] \neq \overline{y}$. Taking into account (3.4), it holds that

$$\begin{aligned} \overline{y} = \alpha y + (1-\alpha)\overline{y} + \alpha(\overline{y} - y) &= [\alpha y + (1-\alpha)\overline{y}] + \alpha(-tr+k) \\ &\in (\alpha y + (1-\alpha)\overline{y}) + (K(y) \setminus \{0\}), \end{aligned}$$

implying the contradicting fact that $\overline{y}$ is not a minimal element of A. $\square$

The next proposition shows hypotheses, under which we can guarantee that, for $y \in A$, $r \in K(y) \setminus \{0\}$, $k \in K(y)$, the element $\overline{y} := y + \psi(y,r)r - k \in A$ is a weakly minimal point.

Proposition 3.5. *Let $A \subset Y$, $K : Y \rightrightarrows Y$, $y \in A$, $r \in K(y) \setminus \{0\}$ and $\overline{y} := y + r\psi(y,r) - k$ for some $k \in K(y)$. Assume that*

(a) *For all $\overline{r} \in K(y) \setminus \{0\}$, $\mathrm{int}(K(y-\overline{r})) \subseteq \mathrm{int}(K(y))$.*

(b) *$K(y)$ is a pointed, closed and convex cone for each $y \in Y$.*

If $\overline{y} \in A$, then $\overline{y} \in \mathrm{WMin}(A, K(\cdot))$.

Proof. First, note that as $K(y)$ is pointed and $y \in A$, taking into account Proposition 3.2(i), it holds that:

$$\psi(y,r) \leq 0.$$

Since $K(y)$ is a convex cone, we have that

$$-\psi(y,r)r + k \in K(y) \setminus \{0\} \text{ for } r, k \in K(y).$$

Assume that $\overline{y} = y + r\psi(y,r) - k$ is not a weakly minimal point. Then there is $\hat{y} \in A$ such that

$$\overline{y} - \hat{y} \in \mathrm{int}(K(\overline{y})). \tag{3.5}$$

Then, as $-\psi(y,r)r + k \in K(y) \setminus \{0\}$, by (a) the following holds

$$\mathrm{int}(K(y - (-r\psi(y,r) + k))) \subseteq \mathrm{int}(K(y)).$$

This means that

$$\mathrm{int}(K(\overline{y})) \subseteq \mathrm{int}(K(y)). \tag{3.6}$$

From (3.5) and (3.6) , we get $\overline{y} - \hat{y} \in \mathrm{int}(K(y))$. So, we conclude for some $\varepsilon > 0$ small enough

$$\overline{y} - \hat{y} - \varepsilon r \in K(y).$$

Substituting $\overline{y}$ and rearranging the expression, we obtain:

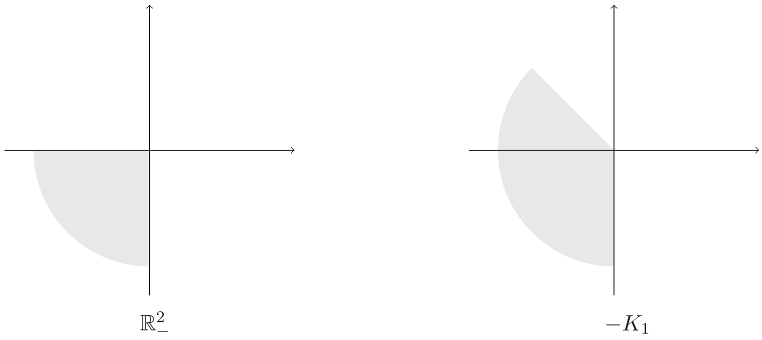

FIGURE 4. The cones $\mathbb{R}^2_-$ and $-K_1$ defining the domination map K at Example 3.7.

$$y + [\psi(y, r) - \epsilon]r \in \hat{y} + K(y) \subseteq A + K(y).$$

Hence $\psi(y, r) \leq \psi(y, r) - \epsilon$, which is impossible. □

Remark 3.6. The proof of the previous proposition relies on the fulfillment of the following two conditions

- $0 \in K(y)$.
- $\forall \lambda \geq 0 : \ \lambda r + k \in K(y)$.

Since the vectors r, k are not known in advance, these two assumptions have to be required for all $r, k \in K(y)$. Then the hypotheses are evidently fulfilled if and only if $K(y)$ is a convex cone. So, unless the point y is given, this sufficient condition has not too much practical use for variable domination mapping $K : Y \rightrightarrows Y$ which are not cone-valued.

Although the hypotheses of the previous proposition are strong, there are non constant maps fulfilling (a) in Proposition 3.5, as is shown at the next example.

Example 3.7. Let $K_1 := \{z \in \mathbb{R}^2 : z_2 \geq -z_1, z_1 \geq 0\}$ and define the domination map, see Figure 4, by:

$$K(y) := \begin{cases} \mathbb{R}^2_-, & \text{if } y \in \mathbb{R}^2_+, \\ -K_1, & \text{otherwise.} \end{cases}$$

If $y \in \mathbb{R}^2_+$, then $K(y) = \mathbb{R}^2_-$. We get $K(y - r) = \mathbb{R}^2_-$ for all $r \in \mathbb{R}^2_- \setminus \{0\}$, since $y - r \in \mathbb{R}^2_+$. This yields $\mathrm{int}(K(y - r)) \subseteq \mathrm{int}(K(y))$ for all $r \in K(y) \setminus \{0\}$.
Furthermore, if $y \notin \mathbb{R}^2_+$, then $K(y) = -K_1$. Since $\mathbb{R}^2_- \subset -K_1$, it holds that $\mathrm{int}(\mathbb{R}^2_-) \subseteq \mathrm{int}(-K_1)$. This yields $\mathrm{int}(K(y - r)) \subseteq \mathrm{int}(-K_1) = \mathrm{int}(K(y))$ for all $r \in K(y) \setminus \{0\}$.

We have proven the relationships between the zeros of the functional and the minimal elements of A. Now more properties will be studied.

3.2. **Algebraic and topological properties of the scalarizing functional.** In this part, we will discuss the continuity, the convexity and the monotonicity of the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ defined in (3.1). These properties are important for deriving the algorithm in Section 4. First, the lower semi-continuity and upper semi-continuity of ψ are analyzed. As usual, Y is a Banach space.

Recall that a given functional $f : Y \to \overline{\mathbb{R}}$ is **lower semi-continuous** (lsc) (**upper semi-continuous** (usc), respectively) at $y^* \in Y$ with $f(y^*)$ is finite, if $\liminf_{y\to y^*} f(y) \geq f(y^*)$, $(\limsup_{y\to y^*} f(y) \leq f(y^*)$, respectively).

Proposition 3.8. *Assume that for all $y \in Y$, $K(y)$ is a pointed set and $A + K(y)$ is convex. Let $(\overline{y}, \overline{r}) \in A \times [Y \setminus \{0\}]$ be fixed and suppose that the mapping $A+K(y)$ is closed at $\overline{y}$ in the sense of Definition 1.1, and that $\psi(\overline{y}, \overline{r})$ is finite. Then the functional $\psi(y, r) : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ is lsc at $(\overline{y}, \overline{r})$.*

Proof. Let $(\overline{y}, \overline{r}) \in A \times Y$. Suppose that $\psi(\overline{y}, \overline{r})$ is finite, but ψ is not lsc at $(\overline{y}, \overline{r})$. Then, there are sequences $\{y_n\}$, $y_n \in Y$, $\{r_n\}$, $r_n \in Y$ and a real number $\varepsilon > 0$ such that $(y_n, r_n) \to (\overline{y}, \overline{r})$ and

$$\psi(y_n, r_n) < \psi(\overline{y}, \overline{r}) - \varepsilon.$$

This means that there is a real number t_n such that

$$t_n < \psi(\overline{y}, \overline{r}) - \varepsilon \tag{3.7}$$

and

$$y_n + t_n r_n \in A + K(y_n).$$

Since $\overline{y} \in A$, it holds that $\psi(\overline{y}, \overline{r}) \leq 0$. So, by (3.7), we get:

$$t_n < \psi(\overline{y}, \overline{r}) - \varepsilon < 0. \tag{3.8}$$

Defining $\alpha_n := \frac{\psi(\overline{y},\overline{r})-\varepsilon}{t_n}$, by (3.8), it is clear that $\alpha \in (0, 1)$.

Furthermore, it holds that $0 \in K(y)$, because $K(y)$ is pointed. Therefore, $\overline{y} \in A$ implies that $\overline{y} \in A+K(y)$ for all $y \in Y$. So, the convexity of $A+K(y)$ for all $y \in Y$, leads us to

$$\alpha_n(y_n + t_n r_n) + (1 - \alpha_n)\overline{y} \in A + K(y_n).$$

Rearranging the terms, we get

$$\alpha_n y_n + [\psi(\overline{y}, \overline{r}) - \varepsilon]r_n + (1 - \alpha_n)\overline{y} \in A + K(y_n).$$

Taking into account that

$$\lim_{n\to+\infty} \alpha_n y_n + (1 - \alpha_n)\overline{y} = \overline{y},$$

$$\lim_{n\to+\infty}[\psi(\overline{y}, \overline{r}) - \varepsilon]r_n = [\psi(\overline{y}, \overline{r}) - \varepsilon]\overline{r}$$

and that $A + K(y)$ is a closed set, we obtain

$$\overline{y} + (\psi(\overline{y}, \overline{r}) - \varepsilon)\overline{r} \in A + K(\overline{y}),$$

contradicting the definition of ψ. So, ψ is lsc at $(\overline{y}, \overline{r})$. □

Remark 3.9. The convexity of $A + K(y)$ can be substituted by the assumption that there exist a bounded set U and a neighborhood V of $\overline{y}$ such that for all $y \in V$, $A + K(y) \subset U$. We again consider a sequence of the type $\{y_n + t_n r_n\}$, $y_n + t_n r_n \in A + K(y_n)$, where $t_n = \psi(\overline{y}, \overline{r}) - \varepsilon_n$. By (3.7), it is clear that $\varepsilon_n \geq \varepsilon > 0$. Since

$A + K(y)$ is bounded, so is ε_n and hence, after taking a convergent subsequence it holds that $\varepsilon_n \to \varepsilon^* \geq \varepsilon > 0$. Taking the limit, we have

$$\overline{y} + (\psi(\overline{y}, \overline{r}) - \epsilon)\overline{r} \in A + K(\overline{y}),$$

contradicting (3.1).

Remark 3.10. For a fixed element $\overline{y} \in Y$, the lsc of $\psi(\overline{y}, r)$ as a functional of r in $\overline{r}$ can be easily obtained, if $A + K(y)$ is closed and star-shaped with respect to $\overline{y}$, that is, if $\overline{y} + r \in K(y)$, then $\overline{y} + \alpha r \in K(y)$ for all $\alpha \in [0, 1]$.

Now, the upper semi-continuity of $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ at $(\overline{y}, \overline{r}) \in Y \times [Y \setminus \{0\}]$ is studied.

Proposition 3.11. *Let $Y = \mathbb{R}^n$, $\overline{y} \in Y$ and $\overline{r} \in Y \setminus \{0\}$. Assume that:*

(a) *$A + K(y)$ is a B-lsc map at $\overline{y}$.*
(b) *$A + K(y)$ is a convex set for all $y \in Y$.*
(c) *The functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ (given by (3.1)) is finite at $(\overline{y}, \overline{r})$.*
(d) *There is $\epsilon_0 > 0$ such that $\overline{y} + t\overline{r} \in \operatorname{int}(A + K(\overline{y}))$ for all $t \in (\psi(\overline{y}, \overline{r}), \psi(\overline{y}, \overline{r}) + \epsilon_0)$.*

Then the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ is usc at $(\overline{y}, \overline{r})$.

Proof. On the contrary, suppose that ψ is finite but not usc at $(\overline{y}, \overline{r})$. Then there are a real number $\varepsilon > 0$ and sequences $\{y_n\}$, $y_n \in Y$, $y_n \to \overline{y}$ and $\{r_n\}$, $r_n \in Y$, $r_n \to \overline{r}$, such that

$$\psi(y_n, r_n) > \psi(\overline{y}, \overline{r}) + \varepsilon.$$

Without loss of generality, we assume that $\varepsilon < \varepsilon_0$.

Let $z = \overline{y} + t\overline{r}$, $t \in [\psi(\overline{y}, \overline{r}) + \varepsilon/2, \psi(\overline{y}, \overline{r}) + \varepsilon)$. Because of (d) we have

$$z \in \operatorname{int}(A + K(\overline{y})).$$

Note that for the elements of the sequence $\{z_n\}$ with $z_n := y_n + tr_n$, it holds that $z_n \notin A + K(y_n)$ because otherwise the following (contradicting) inequality would hold:

$$\psi(\overline{y}, \overline{r}) + \varepsilon < \psi(y_n, r_n) \leq t < \psi(\overline{y}, \overline{r}) + \varepsilon.$$

As, by (b), $A + K(y_n)$ is convex and $z_n \notin A + K(y_n)$, by the classical separation argument given in Theorem 2.4.4, [6], there exists a sequence $\{p_n\}$, $p_n \in Y^*$ (under the given assumptions, we have $Y^* = Y$), $p_n \neq 0$, $n = 1, 2, \ldots$ such that

$$\langle p_n, u\rangle \leq \langle p_n, z_n\rangle \text{ for all } u \in A + K(y_n).$$

Without loss of generality, we assume that $\|p_n\| = 1$, and hence that $p_n \to p$.

By the B-lsc of $A + K(\cdot)$ given in (a), if $u \in A + K(\overline{y})$, there is a sequence $\{u_n\}$ with $u_n \to u$, such that $u_n \in A + K(y_n)$. This implies

$$\langle p, u\rangle \leq \langle p, z\rangle \text{ for all } u \in A + K(\overline{y}). \tag{3.9}$$

But since $z \in \operatorname{int}(A + K(\overline{y}))$, for some $\varepsilon > 0$ it holds that $u = z + \varepsilon p \in A + K(\overline{y})$. Hence, by (3.9),

$$\langle p, z\rangle + \varepsilon\|p\|^2 \leq \langle p, z\rangle,$$

and we get $p = 0$, a contradiction to $\|p\| = \lim \|p_n\| = 1$. So, ψ is an usc functional at $(\overline{y}, \overline{r})$. □

Now, we will provide sufficient conditions for the convexity and the monotonicity of the functional ψ, given in (3.1). We start with the following auxiliary lemma, useful for the analysis of Algorithm 1 in Section 4.

Lemma 3.12. *Suppose that Y is a finite dimensional space, $\overline{y} \in Y$ and $K : Y \rightrightarrows Y$, where $K(y)$ is a convex, closed, pointed set, $K(y) \neq \{0\}$ for all $y \in Y$. If K is B-lsc at $\overline{y} \in Y$ and $k \in \operatorname{int}(K(\overline{y}))$, then there is a neighborhood $V(\overline{y})$ of $\overline{y}$ and a positive number δ such that for all $y \in V(\overline{y})$, it holds that $B(k, \delta) \subset \operatorname{int}(K(y))$.*

Proof. We assume that $\dim(Y) = \ell$ and define

$$E := \{e_1, \dots, e_\ell, -e_1, \dots - e_\ell\},$$

where e_i is the vector whose i^{th}-component is 1 and the rest are 0.

Now, let $k \in \operatorname{int}(K(\overline{y}))$. For simplifying the notations, w.l.o.g. we assume that

$$k + e \in K(\overline{y}) \text{ for all } e \in E. \tag{3.10}$$

Fix $\varepsilon > 0$ and consider the open ball $B(k + e, \varepsilon)$ which, by (3.10), evidently satisfies that

$$K(\overline{y}) \cap B(k + e, \varepsilon) \neq \emptyset.$$

Then, the B-lsc of K at $\overline{y}$ implies that there is a neighborhood $V^e_\varepsilon(\overline{y})$ of $\overline{y}$ such that for all $y \in V^e_\varepsilon(\overline{y})$

$$K(y) \cap B(k + e, \varepsilon) \neq \emptyset. \tag{3.11}$$

Define

$$V_\varepsilon := \cap_{e \in E} V^e_\varepsilon(\overline{y}). \tag{3.12}$$

Since E is finite, V_ε is an open neighborhood of $\overline{y}$ for all $\varepsilon > 0$.

Now, we will prove the existence of $\delta > 0$ small enough, such that for all $y \in Y$ fulfilling $\|y - \overline{y}\| < \delta$, it holds that $k \in \operatorname{int} K(y)$. On the contrary, suppose that there exist sequences $\{y_n\}$, $y_n \in Y$, $y_n \to \overline{y}$ and $\{k_n\}$, $k_n \in Y$, $k_n \to k$, such that $k_n \notin K(y_n)$. W.l.o.g. we can assume that $y_n \in V_{1/n}$, for $n = 1, 2, \dots$ where $V_{1/n}$ is defined in (3.12) for $\varepsilon = 1/n$.

Since $V_{1/n} \subset V^e_{1/n}(\overline{y})$ for all $e \in E$, by (3.11), we get

$$K(y_n) \cap B\left(k + e, \frac{1}{n}\right) \neq \emptyset.$$

Let $\hat{k}_n(e) \in K(y_n) \cap B(k+e, 1/n)$. Since $K(y_n)$ is closed and convex and $k_n \notin K(y_n)$, by [6, Theorem 2.4.4], there exists a sequence $\{p_n\}$, $p_n \in Y^*$ (recall that, since Y is a finite dimensional space, we get $Y^* = Y$), $\|p_n\| = 1$ which separates k_n from $\hat{k}_n(e) \in K(y_n)$. That is, for all $e \in E$:

$$\langle p_n, k_n \rangle > \langle p_n, \hat{k}_n(e) \rangle.$$

Since $\|p_n\| = 1$, there is a convergent subsequence. So, we can assume that $p_n \to p$. Since $\hat{k}_n(e) \to k + e$ for all $e \in E$, we get that

$$\langle p, k \rangle \geq \langle p, k + e \rangle.$$

Taking into account the definition of E and the above inequality, it follows $\langle p, e \rangle = 0$ for all $e \in E$, which implies that $p_i = 0$ for $i = 1, \dots, \ell$. Hence $p = 0$, contradicting the fact that $\|p_n\| = 1$. Therefore, there exists $\delta > 0$ such that for all $y \in B(\overline{y}, \delta)$ it holds that

$$k \in \operatorname{int} K(y).$$

□

Now, the convexity of the functional ψ is analyzed.

Proposition 3.13. *Let $A \subset Y$ be a non-empty and closed set and $K : Y \rightrightarrows Y$ the domination map, where $K(y)$ is a pointed, convex and closed set, $K \neq \{0\}$ for all $y \in Y$. Furthermore, consider the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$, defined by (3.1).*

(i) *Suppose that A is a convex set and that $\alpha K(y^1) + (1-\alpha)K(y^2) \subseteq K(\alpha y^1 + (1-\alpha)y^2)$ for all $\alpha \in [0,1]$ and for all $y^1, y^2 \in Y$. Fix $\overline{r} \in Y \setminus \{0\}$ and assume that $\psi(y, \overline{r}) < +\infty$ for all $y \in Y$. If A is convex, then $\psi(\cdot, \overline{r})$ is a convex functional.*

(ii) *Assume that A is a convex set. For all fixed elements $y \in A$, $\psi(y, \cdot)$ is a quasi-convex functional of r ($r \in K(y) \setminus \{0\}$).*

Proof. (i) Fix $y^1, y^2 \in Y$. Since $\psi(y^i, \overline{r}) < +\infty$, for $i = 1, 2$, there exist sequences $\{t_n^i\}$, $t_n^i \in \mathbb{R}$, $\{a_n^i\}$, $a_n^i \in A$ and $\{k_n^i\}$, $k_n^i \in K(y^i)$ $(i = 1, 2)$ such that $t_n^i \to \psi(y^i, \overline{r})$ and

$$y^1 + t_n^1\overline{r} - k_n^1 = a_n^1,$$
$$y^2 + t_n^2\overline{r} - k_n^2 = a_n^2.$$

So, for all $\alpha \in [0,1]$ it holds that

$$\alpha y^1 + (1-\alpha)y^2 + [\alpha t_n^1 + (1-\alpha)t_n^2]\overline{r} = [\alpha k_n^1 + (1-\alpha)k_n^2] + \alpha a_n^1 + (1-\alpha)a_n^2. \tag{3.13}$$

Since A is convex, we have that $\alpha a_n^1 + (1-\alpha)a_n^2 \in A$. Taking into account that by hypothesis

$$\alpha K(y^i) + (1-\alpha)K(y^2) \subseteq K(\alpha y^1 + (1-\alpha)y^2),$$

we obtain $[\alpha k_n^1 + (1-\alpha)k_n^2] \in K(\alpha y^1 + (1-\alpha)y^2)$. This together with (3.13) imply that

$$\alpha y^1 + (1-\alpha)y^2 + [\alpha t_n^1 + (1-\alpha)t_n^2]\overline{r} \in A + K(\alpha y^1 + (1-\alpha)y^2).$$

Therefore,

$$\psi(\alpha y^1 + (1-\alpha)y^2, \overline{r}) \le [\alpha t_n^1 + (1-\alpha)t_n^2].$$

So, for $n \to +\infty$, we obtain

$$\psi(\alpha y^1 + (1-\alpha)y^2, \overline{r}) \le \alpha\psi(y^1, \overline{r}) + (1-\alpha)\psi(y^2, \overline{r}).$$

Hence, the convexity holds true.

(ii) Now, we consider a fixed element $y \in A$ and $r^1, r^2 \in Y \setminus \{0\}$ arbitrarily chosen. Since $K(y)$ is pointed and $y \in A$, it holds for all $r \in Y$ that

$$\psi(y, r) \le 0, \tag{3.14}$$

as already shown in Proposition 3.2. So,

$$\psi(y, r^1), \psi(y, r^2) < +\infty.$$

By the definition of ψ, this means that there exist two sequences $\{t_n^1\} \in \mathbb{R}$ and $\{t_n^2\} \in \mathbb{R}$ such that $t_n^i \to \psi(y, r^i)$, $i = 1, 2$.

If $\psi(y, r^i) = 0$ holds for some i, $i = 1, 2$, we obtain

$$\psi(y, \alpha r^1 + (1-\alpha)r^2) \leq 0 = \max\{\psi(y, r^1), \psi(y, r^2)\}$$

using (3.14).

Taking into account this inequality, we assume that $\psi(y, r^1), \psi(y, r^2) < 0$. and, hence, w.l.o.g that $t_n^i < 0$, for $i = 1, 2$. By the convexity of A and $K(y)$ for all $y \in A$, we get that $A + K(y)$ is also a convex set for all $y \in A$. Since $y + t_n^i r^i \in A + K(y)$ $(i = 1, 2)$ and $y \in A + K(y)$, we obtain:

$$y + \alpha t_n^i r^i \in A + K(y) \text{ for all } \alpha \in [0, 1].$$

Recalling that $t_n^i < 0$, $i = 1, 2$, it holds that

$$\frac{\max\{t_n^1, t_n^2\}}{t_n^i} \in [0, 1].$$

So, for $i = 1, 2$ and for all $y \in A$ we have

$$y + \max\{t_n^1, t_n^2\} r^i \in A + K(y).$$

Again, by the convexity of $A + K(y)$ for all $y \in A$, we get

$$y + \max\{t_n^1, t_n^2\}[\alpha r^1 + (1-\alpha)r^2] \in A + K(y)$$

for all $\alpha \in [0, 1]$. Then, $\psi(y, \alpha r^1 + (1-\alpha)r^2) \leq \max\{t_n^1, t_n^2\}$ and, taking limits as $n \to +\infty$, we obtain

$$\psi(y, \alpha r^1 + (1-\alpha)r^2) \leq \max\{\psi(y, r^1), \psi(y, r^2)\},$$

as desired. □

Remark 3.14. If $K(y)$ is a cone for all $y \in Y$, the hypothesis

$$\alpha K(y^1) + (1-\alpha)K(y^2) \subseteq K(\alpha y^1 + (1-\alpha)y^2)$$

supposed in Proposition 3.13(i) holds if and only if $K(y) = K$ for all $y \in Y$, where $K \subset Y$ is a (fixed) cone.

We end this section with the study of the homogeneity and the monotonicity of $\psi(\cdot, \overline{r})$ as a functional of $y \in Y$, for fixed $\overline{r} \in Y \setminus \{0\}$.

Definition 3.15. We say that $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ is a **monotone functional in** $y \in Y$ if for all $k \in K(y)$, the inequality

$$\psi(y + k, r) \leq \psi(y, r)$$

is satisfied for all $r \in Y \setminus \{0\}$.

Suppose that for all $y \in Y$, $\text{int}(K(y)) \neq \emptyset$. The functional ψ is called **strictly monotone in** $y \in Y$ if for all $r \in Y \setminus \{0\}$ and $k \in \text{int}(K(y))$, it holds that

$$\psi(y + k, r) < \psi(y, r).$$

Proposition 3.16. *Consider a non-empty and closed set $A \subset Y$, the set-valued map $K : Y \rightrightarrows Y$ and the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ defined by* (3.1).

(i) *Let A be a cone. If for all $y \in Y$ and $\lambda > 0$, $K(\lambda y) = \lambda K(y)$, then for all $\overline{r} \in Y \setminus \{0\}$ such that $\psi(y, \overline{r}) < +\infty$ for all $y \in Y$, it holds that*

$$\psi(\lambda y, \overline{r}) = \lambda \psi(y, \overline{r})$$

for all $\lambda > 0$.

(ii) *Suppose that ψ is finite-valued and that for all $y \in Y$ and for all $k \in K(y)$, $k + K(y) \subseteq K(y + k)$. Then, ψ is a monotone functional in y.*

(iii) *Furthermore, assume that* $\mathrm{int}(K(y)) \neq \emptyset$ *and $r \in K(y) \setminus \{0\}$ for all $y \in Y$. Suppose that the assumptions of (ii) are satisfied and in addition for all $y \in Y$, $k \in K(y)$, $\lambda \geq 0$, $K(y) \subseteq K(y + \lambda k)$. Then, $\psi(\cdot, r)$ is strictly monotone in y.*

(iv) *Suppose that ψ is finite-valued. For all $y \in Y$, assume that $K(y+\lambda\overline{r}) = K(y)$ for all $\overline{r} \in K(y) \setminus \{0\}, \lambda \in \mathbb{R}$. Then, the translation invariance*

$$\psi(y + \lambda\overline{r}, \overline{r}) = \psi(y, \overline{r}) - \lambda$$

is satisfied for all $y \in Y$, $\overline{r} \in K(y) \setminus \{0\}$ and $\lambda \in \mathbb{R}$.

Proof. (i) First, suppose that $\psi(y, \overline{r}) \in \mathbb{R}$. Furthermore, assume that $\lambda > 0$. By the definition of ψ, there exist sequences $\{\varepsilon_n\}$, $\varepsilon_n \in \mathbb{R}$, $\{y_n\}$, $y_n \in A$ and $\{k_n\}$, $k_n \in K(y)$ such that $\varepsilon_n \to 0$, $y = y_n - [\psi(y, \overline{r}) + \varepsilon_n]\overline{r} + k_n$. Multiplying by $\lambda > 0$, it holds that

$$\lambda y = \lambda y_n - \lambda[\psi(y, \overline{r}) + \varepsilon_n]\overline{r} + \lambda k_n.$$

But the assumptions guarantee that $\lambda k_n \in K(\lambda y)$ and $\lambda y_n \in A$, hence $\psi(\lambda y, \overline{r}) \leq \lambda[\psi(y, \overline{r}) + \varepsilon_n]$. Taking limits when $n \to +\infty$, it follows that

$$\psi(\lambda y, \overline{r}) \leq \lambda \psi(y, \overline{r}). \tag{3.15}$$

On the other hand, for $\lambda > 0$ we consider sequences $\{\varepsilon_n^\lambda\}$, $\varepsilon_n^\lambda \in \mathbb{R}$, $\varepsilon_n^\lambda \to 0$, $\{k_n^\lambda\}$, $k_n^\lambda \in K(\lambda y)$ and $\{y_n^\lambda\}$, $y_n^\lambda \in A$ such that:

$$\lambda y = y_n^\lambda - [\psi(\lambda y, \overline{r}) + \varepsilon_n^\lambda]\overline{r} + k_n^\lambda.$$

Dividing by $\lambda > 0$,

$$y = \left[\frac{y_n^\lambda}{\lambda} - \frac{[\psi(\lambda y, \overline{r}) + \varepsilon_n^\lambda]}{\lambda}\overline{r} + \frac{k_n^\lambda}{\lambda}\right]. \tag{3.16}$$

The facts $\frac{k_n^\lambda}{\lambda} \in K(y)$ and $\frac{y_n^\lambda}{\lambda} \in A$, together with (3.16) yield to

$$\psi(y, \overline{r}) \leq \frac{\psi(\lambda y, \overline{r})}{\lambda} + \frac{\varepsilon_n^\lambda}{\lambda}.$$

So, for $n \to +\infty$

$$\psi(y, \overline{r}) \leq \frac{\psi(\lambda y, \overline{r})}{\lambda} \tag{3.17}$$

Taking into account (3.15) and (3.17), we get

$$\psi(\lambda y, r) = \lambda \psi(y, \overline{r}),$$

for $\lambda > 0$.

Now, if $y \in Y$ is such that $\psi(y, \overline{r}) = -\infty$, there exist sequences $\{t_n\}$, $t_n \in \mathbb{R}$, $\{y_n\}$, $y_n \in A$, and $\{k_n\}$, $k_n \in K(y)$ such that $t_n \to -\infty$ and

$$y = y_n - t_n\overline{r} + k_n.$$

As before, the multiplication by $\lambda > 0$ lead us to $\lambda y = \lambda y_n - \lambda t_n \overline{r} + \lambda k_n$, where $\lambda y_n \in A$ and $\lambda k_n \in K(\lambda y)$. Taking into account that $t_n \to -\infty$ and $\lambda > 0$ is fixed, it holds that $\lambda t_n \to -\infty$. So, $\psi(\lambda y, \overline{r}) = -\infty$.

(ii) Consider $y \in Y$, $r \in Y \setminus \{0\}$ and $k \in K(y)$. By the definition of ψ, there are sequences $\{t_n\}$, $t_n \in \mathbb{R}$, $t_n \to \psi(y, r)$, $\{y_n\}$, $y_n \in A$ and $\{k_n\}$, $k_n \in K(y)$, such that

$$y + t_n r = y_n + k_n.$$

This yields

$$y + k + t_n r = y_n + k_n + k.$$

Since $k \in K(y)$ implies that $k + K(y) \subseteq K(y + k)$ and $k_n \in K(y)$, it holds that $k_n + k \in K(y + k)$. So,

$$y + k + t_n r \in A + K(y + k)$$

and

$$\psi(y + k, r) \leq t_n.$$

Taking limits in the previous relation, for $n \to +\infty$

$$\psi(y + k, r) \leq \psi(y, r),$$

which means that $\psi(\cdot, k)$ is monotone in y, as desired.

(iii) First, we will prove that

$$\psi(y + \lambda\hat{r}, \hat{r}) \leq \psi(y, \hat{r}) - \lambda, \tag{3.18}$$

if $\lambda > 0$, $\hat{r} \in K(y)$ for all $y \in Y$.

Indeed, since ψ is finite-valued, there exist sequences $\{t_n\}$, $t_n \in \mathbb{R}$, $t_n \to \psi(y, r)$, $\{y_n\}$, $y_n \in A$, and $\{k_n\}$, $k_n \in K(y)$ such that:

$$y + t_n\hat{r} = y_n + k_n,$$

or, equivalently,

$$y + \lambda\hat{r} + [t_n - \lambda]\hat{r} = y_n + k_n.$$

Under the assumption $K(y) \subseteq K(y + \lambda\hat{r})$, it follows that

$$y + \lambda\hat{r} + [t_n - \lambda]\hat{r} \in A + K(y + \lambda\hat{r}).$$

This means that

$$\psi(y + \lambda\hat{r}, \hat{r}) \leq t_n - \lambda.$$

Taking limits for $n \to +\infty$, (3.18) is obtained.

Now, take $k \in \operatorname{int}(K(y))$. So, there is $\varepsilon > 0$ such that $k - \varepsilon r \in \operatorname{int}(K(y))$. By (ii), ψ is monotone. This means that

$$\psi(y + k - \varepsilon r, r) \leq \psi(y, r) \tag{3.19}$$

holds.

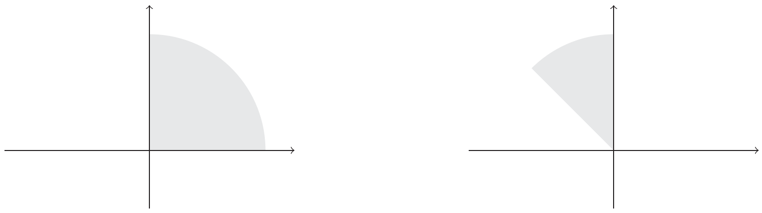

FIGURE 5. The cones $\mathbb{R}^2_+$ and K_1 defining the domination map K in Example 3.19.

Combining (3.18) and the monotonicity condition (3.19), we arrive at

$$\psi(y+k,r) \le \psi(y+k-\varepsilon r, r) - \varepsilon \le \psi(y,r) - \varepsilon < \psi(y,r),$$

which is equivalent to the strict monotonicity of $\psi(\cdot, r)$ in y.

(iv) Under our assumptions, $K(y+\lambda\overline{r}) = K(y)$ for all $y \in Y$, $\overline{r} \in K(y) \setminus \{0\}$ and $\lambda \in \mathbb{R}$. Following the proof of (iii), we obtain a relation analogous to (3.18). That is, for all $y \in Y$, $\overline{r} \in K(y) \setminus \{0\}$ and $\lambda \in \mathbb{R}$, the following inequality holds:

$$\psi(y+\lambda\overline{r}, \overline{r}) \le \psi(y, \overline{r}) - \lambda. \tag{3.20}$$

Then

$$\psi(y-\lambda\overline{r}, \overline{r}) = \psi(y+(-\lambda)\overline{r}, \overline{r}) \le \psi(y,\overline{r}) - (-\lambda) = \psi(y,\overline{r}) + \lambda,$$

and in particular as $y = y + \lambda\overline{r} - \lambda\overline{r}$,

$$\psi(y, \overline{r}) \le \psi(y+\lambda\overline{r}, \overline{r}) + \lambda. \tag{3.21}$$

The combination of this inequality and (3.20) leads to the desired result. □

Remark 3.17. If $K(y)$ is a convex cone for all $y \in Y$, the hypothesis of Proposition 3.16 (ii) is $K(y) \subseteq K(y+r)$ for all $r \in K(y)$. This is equivalent to the hypothesis of (iii). In (iv), the equality between both sets must hold.

Now, we will present non-constant set-valued maps which fulfill the hypotheses of Proposition 3.16. We start with (i).

Example 3.18. Let $T : Y \times [Y \setminus \{0\}] \to \mathbb{R}$ be a bi-linear operator and consider the domination structure given by $K(y) = \{z \in Y : T(y,z) \le 0\}$. Evidently, we get $K(y) = K(\lambda y)$ for all $\lambda > 0$.

Clearly, if $K(y+r) = K(y)$ for all $r \in K(y)$, the conditions supposed at Proposition 3.16 (ii)-(iv) hold. Based on the close relationship between them, the next example presents a domination structure which satisfies this stronger property.

Example 3.19. Let $K_1 := \left\{(r\cos(\theta), r\sin(\theta)) : r \ge 0, \theta \in \left[\frac{\pi}{2}, \frac{5\pi}{4}\right]\right\}$ and

$$K(y) := \begin{cases} \mathbb{R}^2_+, & \text{if } y_1 \ge 0, \\ K_1, & \text{otherwise.} \end{cases}$$

Figure 5 shows the two cones which define the domination given in Example 3.19.

If $y_1 \ge 0$, and $r \in K(y)$, then $r \ge 0$ and, hence, $r_1 + y_1 \ge 0$. So, $K(y+r) = \mathbb{R}^2_+ = K(y)$. In the other case, i.e., if $y_1 < 0$, $r \in K(y)$ implies that $r_1 \le 0$. So, $y_1 + r_1 < 0$ and again $K(y+r) = K(y)$.

Based the properties of the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ presented in this section, we propose an algorithm for computing weakly minimal points of the geometric vector optimization problem under variable domination structure introduced in (2.1) (see Section 4).

4. An algorithm for solving geometric optimization problems under variable domination structure

This part contains an algorithm that can be used for solving geometric vector optimization problems with variable domination structure, see Section 2, i.e., to find an element of $\mathrm{WMin}(A, K(\cdot))$ according to Definition 2.2. In this section we assume $Y = \mathbb{R}^m$. As before, we assume that the domination structure is given by the set-valued map $K : Y \rightrightarrows Y$ and that $K(y)$ is a convex, closed and pointed cone with $\mathrm{int}(K(y)) \neq \emptyset$ for all $y \in Y$. As a measure which determines whether a point is an element of $\mathrm{WMin}(A, K(\cdot))$ or not, we will use the scalarization defined in (3.1) by the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$, $\psi(y, r) = \inf\{t \in \mathbb{R} \ : \ y + tr \in A + K(y)\}$. We define the algorithm as follows (where $\mathbb{Z}$ represents the set of integers):

Algorithm 1

Initial Step Find $y_0 \in A$. $n = 0$.

Iterative Step

- If

$$\inf_{\|r\|=1} \psi(y_n, r) = 0, \tag{4.1}$$

  EXIT, else take r_n such that $\psi(y_n, r_n) < 0$ and $\|r_n\| = 1$. Find $j_n \in \mathbb{Z}, k_n \in K(y_n)$ such that

$$j_n := \sup\{j \in \mathbb{Z} : y_n - 2^j r_n - k_n \in A, k_n \in K(y_n)\}. \tag{4.2}$$

-

$$y_{n+1} := y_n - 2^{j_n} r_n - k_n, \tag{4.3}$$

  $n = n + 1$.

Remark 4.1. Under the assumptions of Proposition 3.2, the functional $\psi : Y \times [Y \setminus \{0\}] \to \overline{\mathbb{R}}$ is finite-valued. So, we suppose in the following that the assumptions of Proposition 3.2 are satisfied. Furthermore, if the hypothesis of Propositions 3.8 and 3.11 are fulfilled for $\overline{y} = y_n$ and for all $\overline{r}$ with $\|\overline{r}\| = 1$, the functional ψ is continuous for all feasible elements of the auxiliary problem (4.1). Then the infimum is attained.

If $\|\cdot\|$ represents the Euclidean norm, the set of feasible solutions of problem (4.1) is not convex. However another norm $\|\cdot\|$ such that $\{r \in Y : \ \|r\| = 1\}$ is convex can be considered. In this very restrictive case, that the assumptions of Proposition 3.13(ii) are fulfilled, the objective functional is quasiconvex.

Let us characterize the limit points of the sequence generated by Algorithm 1.

Proposition 4.2. *Suppose that $Y = \mathbb{R}^m$ and that $K : Y \rightrightarrows Y$ is a set valued map, where $K(y)$ is a pointed, convex and closed cone with $\mathrm{int}(K(y)) \neq \emptyset$ for all $y \in Y$. Furthermore, assume that K is a B-lsc and closed map at all $y \in Y$. If $A \subset Y$*

is a convex closed set with non-empty interior and the sequence $\{y_n\}$ generated by Algorithm 1 converges to $\overline{y}$, then its limit is a weakly minimal point of (2.1)*.*

Proof. Let $\{y_n\}$ be the sequence generated by Algorithm 1 and $\overline{y}$ its limit. Suppose that $\overline{y} \notin \mathrm{WMin}(A, K(\cdot))$. Then there is an element $y^0 \in A$ with $\overline{y} = y^0 + \overline{k}$ for a certain $\overline{k} \in \mathrm{int}(K(\overline{y}))$. We take $\delta > 0$ such that $B(\overline{k}, \delta) \subset \mathrm{int}(K(\overline{y}))$. Since A is a convex set with non-empty interior, there is an element $\hat{y} \in \mathrm{int}(A) \cap B(y^0, \delta)$. So, it holds that

$$\overline{y} - \hat{y} - \overline{k} = y^0 - \hat{y} \in B(0, \delta).$$

Hence, we get

$$\overline{y} - \hat{y} \in B(\overline{k}, \delta) \subset \mathrm{int}\, K(\overline{y}).$$

This means that there is an element $\hat{k} \in \mathrm{int}(K(\overline{y}))$ with $\overline{y} = \hat{y} + \hat{k}$. Because of $\hat{y} \in \mathrm{int}(A)$, we can find $\alpha > 0$ such that

$$B(\hat{y}, \alpha) \subset A. \tag{4.4}$$

We consider

$$\begin{aligned} y_n - (\hat{y} - \alpha r_n) &= (-y_{n+1} + y_n) + (y_{n+1} - \overline{y}) + \overline{y} - \hat{y} + \alpha r_n, && (4.5)\\ &= 2^{j_n} r_n + k_n + (y_{n+1} - \overline{y}) + \hat{k} + \alpha r_n && (4.6)\\ &= (2^{j_n} + \alpha) r_n + \hat{k} + k_n + (y_{n+1} - \overline{y}), && (4.7) \end{aligned}$$

where $\{j_n\}$, $j_n \in \mathbb{Z}$, $\{k_n\}$, $k_n \in K(y_n)$ and $\{r_n\}$, $r_n \in Y$, are the sequences generated by Algorithm 1 for defining y_{n+1}, see (4.3).

Note that $\hat{k} \in \mathrm{int}(K(\overline{y}))$ and K is a B-lsc map at $\overline{y}$. So, by Lemma 3.12, there is a neighborhood V of $\overline{y}$ and a real number $\varepsilon > 0$ such that for all $y \in V$, $B(\hat{k}, \varepsilon) \subset K(y)$. In particular, since $y_n \to \overline{y}$, we get $B(\hat{k}, \varepsilon) \subset K(y_n)$ for n large enough. Taking into account that $y_{n+1} - \overline{y} \to 0$ for n large enough, it holds that

$$\hat{k} + y_{n+1} - \overline{y} \in K(y_n).$$

Since $K(y_n)$ is a convex cone, $\hat{k} + y_{n+1} - \overline{y} \in K(y_n)$ and $k_n \in K(y_n)$ imply $\hat{k} + y_{n+1} - y + k_n \in K(y_n)$. Then we get from (4.7) and (4.4):

$$y_n - (2^{j_n} + \alpha) r_n = (\hat{y} - \alpha r_n) + (\hat{k} + y_{n+1} - \overline{y} + k_n) \in A + K(y_n).$$

Using the definition of j_n in equation (4.2), it holds that

$$2^{j_n} + \alpha \leq 2^{j_n + 1}.$$

So, we get

$$0 < \alpha < 2^{j_n}. \tag{4.8}$$

Note that since $y_n \to \overline{y}$ (so, of course, $y_{n+1} \to \overline{y}$), by (4.3)

$$-2^{j_n} r_n - k_n \to 0. \tag{4.9}$$

Then

$$\lim_{n \to +\infty} \|2^{j_n} r_n + k_n\|^2 = 0,$$

which is equivalent to

$$\lim_{n \to +\infty} (2^{j_n} - \|k_n\|)^2 + 2^{j_n + 1} \|k_n\| \left(\langle r_n, \frac{k_n}{\|k_n\|} \rangle + 1 \right) = 0,$$

taking into account that $\|r_n\| = 1$.
Since both terms are non-negative, we get

$$\lim_{n\to+\infty}(2^{j_n} - \|k_n\|)^2 = \lim_{n\to+\infty} 2^{j_n+1}\|k_n\| \left(\langle r_n, \frac{k_n}{\|k_n\|}\rangle + 1\right) = 0.$$

In particular, the second limit implies that either

$$\left(\left\langle r_n, \frac{k_n}{\|k_n\|}\right\rangle + 1\right) \to 0 \tag{4.10}$$

or

$$2^{j_n+1}\|k_n\| \to 0. \tag{4.11}$$

Suppose that (4.10) holds. Due to $\{r_n\}$ and $\{\frac{k_n}{\|k_n\|}\}$ are bounded, we can consider that $r_n \to r^*$ and $\frac{k_n}{\|k_n\|} \to k^*$. It is clear that $\|r^*\| = \|k^*\| = 1$ and hence that $r^* \neq 0$. This leads us to

$$r^*, k^* \in K(y) \setminus \{0\}$$

taking into account that $K(y)$ is closed.

But (4.10) implies

$$r^* = -k^* \in K(y) \setminus \{0\},$$

which contradicts the pointedness of $K(y)$. So, $2^{j_n}\|k_n\| \to 0$.

Because of

$$\lim_{n\to+\infty}(2^{j_n} - \|k_n\|)^2 = 0,$$

we obtain

$$\lim_{n\to+\infty} 2^{j_n} = \lim_{n\to+\infty} \|k_n\| = 0.$$

Since α is fixed, for n large enough, (4.8) is not fulfilled. Hence y is a weakly minimal point. □

Remark 4.3. Note that the condition that the variable domination mapping $K : Y \rightrightarrows Y$ is cone-valued cannot be relaxed. Indeed, $K(y)+K(y) \subseteq K(y)$ for all $y \in Y$ is needed. Since $K(y)$ is pointed and convex, this implies that $K(y)$ is a convex cone.

We now prove the convergence of the algorithm.

Theorem 4.4. *Suppose that*

(a) $Y = \mathbb{R}^m$.
(b) $K : Y \rightrightarrows Y$ *is a B-lsc closed map such that* $K(y)$ *is a pointed, convex and closed cone with* $\mathrm{int}(K(y)) \neq \emptyset$ *for all* $y \in Y$.
(c) *There is a closed, convex and pointed cone* $\mathcal{K}$ *such that* $K(y) \subset \mathcal{K}$ *for all* $y \in Y$.
(d) $A \subset Y$ *is convex and* $\mathrm{int}(A) \neq \emptyset$.

If the sequence $\{y_n\}$ *generated by Algorithm 1 has an accumulation point* y*, then,* $y_n \to y$ *and* y *is a weakly minimal point of* A *with respect to the domination mapping* K.

Proof. Let $\mathcal{K}$ be a closed convex and pointed cone such that $K(y) \subset \mathcal{K}$ for all $y \in Y$. Note that $\mathcal{K}$ is proper because it is pointed and contains proper cones. As $K(y) \subset \mathcal{K}$, $\{y_n\}$ is a decreasing sequence with respect to $\mathcal{K}$. Since $\{y_n\}$ has a convergent subsequence, it converges. Proposition 4.2 guarantees that the limit is a weakly minimal point. □

Corollary 4.5. *If $A \subset Y$ is bounded and the hypotheses of Theorem 4.4 hold, Algorithm 1 converges.*

Proof. The assertion follows evidently from Theorem 4.4, because $y_n \in A$ and A is bounded. □

Although the algorithm is simple, at each iteration it must determine whether certain points are elements of A or not. So, it can only be recommended in cases when this step is not too expensive. However, in practical cases A may have a more complex structure. If A is the image of C by a certain function $F : C \to Y$, in general there does not exist an efficient algorithm which, for $y \in Y$, establishes whether $y \in A$ or not. In the next section, Algorithm 1 is adapted to this case.

5. Functional vector optimization problems with variable domination structures

In this part we will assume that X and Y are finite dimensional spaces and that the variable domination in Y is defined by a set-valued mapping $K : Y \rightrightarrows Y$, where $K(y)$ is a convex, closed and pointed cone with $\operatorname{int}(K(y)) \neq \emptyset$ for all $y \in Y$. Given a closed subset C of X and the C^1-differentiable function $F : X \to Y$, we consider the optimization problem

$$K(\cdot) - \min F(x) \text{ s.t. } x \in C. \tag{5.1}$$

x^* is a solution of Problem (5.1) with respect to K if for all $x \in C$, $F(x) \notin F(x^*) - (K(F(x^*)) \setminus \{0\})$.

This model is clearly related with (2.1) because it is equivalent to finding the elements of $\operatorname{Min}(F(C), K(\cdot))$. As already pointed out, Algorithm 1 is not an efficient alternative. However, the structure of the set can be used. Suppose that $F \in C^1$ and $v \in C - x$, then $F(x + v)$ can be approximated as follows:

$$F(x + v) \approx F(x) + \nabla F(x)v$$

and the set A as the image of this linear function. That is:

$$A(x) \approx \{F(x) + \nabla F(x)v \ : \ v \in C - x\}.$$

So, we can solve the approximate linearized problem $\operatorname{Min}(A(x), K(\cdot))$ and the weakly minimal elements of $A(x)$ will be used as the next iteration point. The stopping criterion reads $\nabla F(x)v \notin -\operatorname{int}(K(F(x)))$ which means that x is a weakly stationary point. The formal definition is:

Definition 5.1. Let $F \in C^1$. We say that $x \in C$ is a **weakly stationary point** of problem (5.1) if $\nabla F(x)v \notin -\operatorname{int}(K(F(x)))$ for all $v \in C - x$.

As already pointed out in [7], weakly minimal points are also weakly stationary. So, we expect that by solving at each step the linearized problem $\mathrm{WMin}(A(x), K(\cdot))$, Algorithm 1 will compute a sequence which either stops after finitely many steps or the accumulation points of the generated sequence are weakly stationary points. Given $x \in C$, for $A = A(x)$, the scalarizing functional is

$$\psi(F(x), r) = \inf\{t : tr \in \nabla F(x)v + K(F(x)), v \in C - x\}. \tag{5.2}$$

It is clear that the linear term $\nabla F(x)$ may lead to the unboundedness of ψ. This obstacle can be overcome by adding $\|v\|^2/2$. However, even with this new term, the resulting functional is not convex in r. As proven in Proposition 3.13, $\psi(F(x), \cdot)$ is only a quasi-convex functional. So, the solution of the approximate problem $\inf_{v \in C-x} \|v\|^2/2 + \inf_{\|r\|=1} \psi(y, r)$ is not simple. Moreover, a point of the form $F(x_n) - k = F(x_{k+1})$ for some $k \in K(F(x_n))$ must be found. That is why we will consider domination maps $K : Y \rightrightarrows Y$ such that it is easy to verify that $k \in K(y)$ and the scalarizing functional (5.2) enjoys good properties. In particular, we will always assume that

(H1) $l : \mathbb{R}^m \times [\mathbb{R}^m \setminus \{0\}] \to \mathbb{R}$ is a continuous functional.
(H2) For all $y \in \mathbb{R}^m$, $l(y, \cdot)$ is a convex and positively homogeneous functional of z.
(H3) The set $\{l(y, z) \ : \ y \in \mathbb{R}^m, \ z \in \mathbb{R}^m, \ \|z\| = 1\}$ is bounded.
(H4) The cone $K(y)$ is defined as

$$K(y) := \{z \in \mathbb{R}^m \setminus \{0\} \ : \ l(y, z) \leq 0\}$$

and

$$\mathrm{int}(K(y)) = \{z \in \mathbb{R}^m \setminus \{0\} \ : \ l(y, z) < 0\} \neq \emptyset \text{ for all } y \in Y.$$

(H5) $F \in C^1$.

Note that K is a closed map since $l(y, z)$ is a continuous functional.

Given this representation of the cone, it is easier to use $l(y, z)$ to define the auxiliary functional

$$\theta(x, v) := l(F(x), -\nabla F(x)v) \qquad (x, v \in X).$$

If C is unbounded, $\theta(x, v)$ can be unbounded. In this case $\inf_{v \in C-x} \theta(x, v) = -\infty$ and the iterative step will be unsuccessful. As before, consider the problem

$$\inf_{v \in C-x} \frac{\|v\|^2}{2} + \theta(x, v).$$

In this framework, the algorithm for computing weakly stationary points of problem (5.1) is formulated as follows:

Algorithm 2

Initial Step Find $x_0 \in C$. $n = 0$, $\sigma \in (0, 1)$.

Iterative Step

- Find a solution v_n of the problem

$$\arg \min_{v \in C-x_n} \|v\|^2/2 + l(F(x_n), -\nabla F(x_n)v). \tag{P_n}$$

- If $v_n = 0$, EXIT, else find $j_n \in \mathbb{Z}$ such that

$$j_n := \sup\{j \in \mathbb{Z} \ : \ F(x_n) + \sigma 2^j \nabla F(x_n) v_n - F(x_n + 2^j v_n) \in K(F(x_n))\}.$$

- $x_{n+1} := x_n + 2^{j_n} v_n$, $n = n + 1$.

Now, we will show that Algorithm 2 is well-defined.

Proposition 5.2. (i) *The problem (P_n) has a unique minimizer v_n.*

(ii) *$v_n = 0$ is a unique solution of (P_n) if and only if x_n is a weakly stationary point.*

(iii) *There exists $M > 0$ such that $\|v_n\| \le M \|\nabla F(x_n)\|$ for all $n = 1, 2, \ldots$.*

(iv) *If $v_n \neq 0$, then there exists $j \in \mathbb{Z}$ such that $F(x_n) + 2^j \nabla F(x_n) v_n - F(x_n + 2^j v_n) \in K(F(x_n))$.*

Proof. (i) By the positive homogeneity of l, $\|v\|^2/2 + l(F(x_n), -\nabla F(x_n)v) \to +\infty$ if $\|v\| \to +\infty$. This implies the existence of a solution of (P_n). On the other hand, $\|v\|^2/2$ is a strictly convex functional and $l(F(x_n), -\nabla F(x_n)v)$ is convex in v due to (H2). So, the objective function of (P_n) is strictly convex and, therefore, the minimum is also unique.

(ii) It is clear that 0 is a feasible solution of problem (P_n), hence

$$\lambda_n := \min_{v \in C - x_n} \|v\|^2/2 + l(F(x_n), -\nabla F(x_n)v) \le 0. \tag{5.3}$$

(a) If x_n is a weakly stationary point, it holds that $-\nabla F(x_n)v \notin \operatorname{int}(K(F(x_n)))$ for all $v \in C - x_n$, which means that

$$l(F(x_n), -\nabla F(x_n)v) \ge 0 \text{ for all } v \in C - x_n.$$

Then, for all $v \in C - x_n$ it holds that

$$\|v\|^2/2 + l(F(x_n), -\nabla F(x_n)v) \ge 0.$$

Combining this inequality with (5.3), we obtain $\lambda_n = 0$.

(b) Now, we will prove that $\lambda_n < 0$, if x_n is not a weakly stationary point.

Let $v \in C - x_n$ be such that $-\nabla F(x_n)v \in \operatorname{int}(K(F(x_n)))$. By (H4), it holds that

$$l(F(x_n), -\nabla F(x_n)v) < 0.$$

Recalling that l is a positively homogeneous map, for some $\alpha > 0$ small enough, it holds that

$$\|\alpha v\|^2/2 + l(F(x_n), -\nabla F(x_n)\alpha v) = \alpha^2 \|v\|^2/2 + \alpha l(F(x_n), -\nabla F(x_n)v) < 0.$$

Since C is convex, we have $\alpha v \in C - x_n$ for all $\alpha \in [0, 1]$. Then, as desired

$$\lambda_n \le \alpha \|v\|^2/2 + l(F(x_n), -\nabla F(x_n)\alpha v) < 0.$$

By (a) and (b), it holds that $\lambda_n = 0$ if and only if x_n is a weakly stationary point. Since the objective function of (P_n) evaluated at $v = 0$ coincides with the minimal value and, by (i), (P_n) has a unique solution, we obtain that $v_n = 0$ is its unique solution if and only if x_n is a weakly stationary point.

(iii) By (5.3), we get for a solution v_n of (P_n)

$$\|v_n\|^2/2 \le -l(F(x_n), -\nabla F(x_n)v_n) = |-l(F(x_n), -\nabla F(x_n)v_n)|.$$

Again, since l is a continuous, positively homogeneous functional, we have

$$\|v_n\|^2/2 \le |-l(F(x_n), -\nabla F(x_n)v_n)| \le \max_{\|r\|=1} |l(F(x_n), r)| \|\nabla F(x_n)v_n\|.$$

By (H3), $l(F(x_n), r)$ is bounded for all r with $\|r\| = 1$. Hence, for

$$M = 2 \max_{\|r\|=1} |l(F(x_n), r)|,$$

we obtain the relation $\|v_n\|^2 \le M\|\nabla F(x_n)v_n\| \le M\|\nabla F(x_n)\|\|v_n\|$ and, therefore,

$$\|v_n\| \le M\|\nabla F(x_n)\|.$$

(iv) $v_n \neq 0$ means that $\nabla F(x_n)v_n \in -\operatorname{int}(K(F(x_n)))$. Considering the Taylor expansion of $F(x_n + 2^j v_n)$, we get

$$F(x_n) + 2^j \nabla F(x_n)v_n + o(2^j) = F(x_n + 2^j v_n)$$

and, equivalently,

$$F(x_n) + 2^j \sigma \nabla F(x_n)v_n - F(x_n + 2^j v_n) = -2^j(1-\sigma)\nabla F(x_n)v_n + o(2^j).$$

Taking into account $-\nabla F(x_n)v_n \in \operatorname{int}(K(F(x_n))$ and $\sigma < 1$, we get

$$-(1-\sigma)\nabla F(x_n)v_n \in \operatorname{int}(K(F(x_n))).$$

So, there is $j \in \mathbb{Z}$ such that

$$-(1-\sigma)\nabla F(x_n)v_n + \frac{o(2^j)}{2^j} \in \operatorname{int}(K(F(x_n))) \subset K(F(x_n)),$$

which means that

$$-2^j(1-\sigma)\nabla F(x_n)v_n + o(2^j) \in \operatorname{int}(K(F(x_n))) \subset K(F(x_n)).$$

□

Now, a simple sufficient condition for the convergence of $\{F(x_n)\}$ is obtained.

Proposition 5.3. *Suppose that the conditions [H1]-[H5] hold and that there exists a closed, convex and pointed cone $\mathcal{K} \subset Y$, such that $K(F(x)) \subset \mathcal{K}$ for all $x \in C$. If x^* is an accumulation point of the sequence $\{x_n\}$ generated by Algorithm 2, then $F(x_n) \to F(x^*)$.*

Proof. First, note that $F(x^*)$ is a limit point of $\{F(x_n)\}$ since F is a continuous function and x^* is an accumulation point of the sequence $\{x_n\}$. By the definition of Algorithm 2, $F(x_n) + \sigma 2^{j_n} \nabla F(x_n)v_n - F(x_{n+1})$ and $-\nabla F(x_n)v_n$ are elements of $K(F(x_n))$. In particular, since $K(F(x_n))$ is a convex cone, it holds that

$$F(x_n) + (\sigma 2^{j_n} - \alpha)\nabla F(x_n)v_n - F(x_{n+1}) \in K(F(x_n)) \tag{5.4}$$

for all $\alpha > 0$. Taking $\alpha := \sigma 2^{j_n} \ge 0$, we obtain

$$F(x_n) - F(x_{n+1}) \in K(F(x_n)) \subset \mathcal{K}.$$

This implies that $\{F(x_n)\}$ is a decreasing sequence with an accumulation point. Due to the fact that $\mathcal{K}$ is a closed, convex, pointed cone and $\mathcal{K} \neq \{0\}$, since $\{0\} \neq K(y) \subset \mathcal{K}$ for all $y \in Y$, it holds that

$$F(x_n) \to F(x^*).$$

□

Now, we will analyze the properties of the accumulation points of the sequence generated by Algorithm 2. We start with some preliminary results.

Lemma 5.4. *Suppose that $\{x_n\}$ is a subsequence of points generated by Algorithm 2 which converges to x^*. Then the solution of the associated problem (P_n) has a convergent subsequence $\{v_{n_k}\}$, and if $v_{n_k} \to v^*$, $v^* \in C - x^*$. Moreover, if $l(F(x^*), -\nabla F(x^*)v^*) = 0$, then x^* is a weakly stationary point of F over C.*

Proof. By Proposition 5.2, we get that $\|v_n\|$ is bounded since x_n converges and $F \in C^1$. Hence, it has an accumulation point. For simplicity, we will assume that the whole sequence $\{v_n\}$ converges to v^*. Since C is closed, the mapping $\mathcal{C} : X \rightrightarrows X$, defined by $\mathcal{C}(x) := C - x$ is closed. Then, due to $x_n \to x^*$, $v_n \to v^*$ and $v_n \in C - x_n$, it holds that $v^* \in C - x^*$.

For the second part, note that, as $x_n \to x^*$ and $v_n \to v^*$, by the continuity of $\nabla F(x)$ we get

$$\lim_{n\to+\infty} l(F(x_n), -\nabla F(x_n)v_n) = l(F(x^*), -\nabla F(x^*)v^*) = 0.$$

Suppose that x^* is not a weakly stationary point. Then there exists $v \in C - x^*$ such that

$$l(F(x^*), -\nabla F(x^*)v) < 0.$$

Using the same ideas as in the proof of Proposition 5.2(iii) we can assume that

$$\frac{\|v\|^2}{2} + l(F(x^*), -\nabla F(x^*)v) < 0.$$

In particular, consider $(v+(x-x_n))$, which is evidently an element of $C-x_n$. Since v_n is the solution of problem (P_n), we get

$$\frac{\|v_n\|^2}{2} + l(F(x_n), -\nabla F(x_n)v_n) \leq \frac{\|v + (x - x_n)\|^2}{2} + l(F(x_n), -\nabla F(x_n) + (x - x_n)).$$

Taking limits for $n \to +\infty$, we conclude

$$0 \leq \frac{\|v^*\|^2}{2} + l(F(x^*), -\nabla F(x^*)v^*) \leq \frac{\|v\|^2}{2} + l(F(x^*), -\nabla F(x^*)v) < 0.$$

This contradiction implies that x^* is a weakly stationary point. □

Corollary 5.5. *Let $\{x_n\}$ be a subsequence of points generated by Algorithm 2 converging to x^* and v_n be the solution of (P_n). Suppose that $v_n \to v^*$. If $\nabla F(x^*)v^* = 0$, then x^* is a weakly stationary point of F with respect to C.*

Proof. Note that $l(F(x), 0) = 0$. Therefore, $l(F(x^*), -\nabla F(x^*)v^*) = 0$. So, the result follows directly from Lemma 5.4. □

Theorem 5.6. *Let us assume that [H1]-[H5] are satisfied and that $K(F(x)) \subset \mathcal{K}$ for all $x \in C$, where $\mathcal{K}$ is a closed, convex and pointed cone in Y. Then every accumulation point of the sequence generated by Algorithm 2 is a weakly stationary point.*

Proof. Let $\{x_n\}$ be a subsequence generated by Algorithm 2, such that $x_n \to x^*$. As already proven in Proposition 5.3, $F(x_n) \to F(x^*)$. Moreover, by Lemma 5.4, without loss of generality, we can assume that $v_n \to v^*$. On the other hand, it holds that

$$F(x_n) + \sigma 2^{j_n} \nabla F(x_n) v_n - F(x_{n+1}) \in K(F(x_n)). \tag{5.5}$$

Now, we consider two cases:

Case 1: If $\{2^{j_n}\}$ is upper bounded, then the sequence $\{2^{j_n}\}$ is bounded since it is non-negative. We consider a convergent subsequence $\{2^{j_n}\} \to t^*$, and taking the limit for $n \to +\infty$ in (5.5), we obtain

$$t^* \nabla F(x^*) v^* \in K(F(x^*)). \tag{5.6}$$

By the definition of Algorithm 2, it holds that

$$-\nabla F(x_n) v_n \in K(F(x_n))$$

and, since $K(F(x_n))$ is a cone,

$$-2^{j_n} \nabla F(x_n) v_n \in K(F(x_n)).$$

So, when $n \to +\infty$

$$-t^* \nabla F(x^*) v^* \in K(F(x^*)).$$

Taking into account (5.6) and the last inclusion, we get

$$-t^* \nabla F(x^*) v^* \in K(F(x^*)) \cap -K(F(x^*)).$$

Since $K(F(x^*))$ is pointed, it follows

$$t^* \nabla F(x^*) v^* = 0.$$

Here we consider again two cases:

Case (1.a) If $t^* > 0$, then $\nabla F(x^*) v^* = 0$. By Corollary 5.5, x^* is a weakly stationary point.

Case (1.b) Now, we assume that $2^{j_n} \to 0$ and fix $q \in \mathbb{N}$. Then there exists $k_q \in \mathbb{N}$, such that for all $n > k_q$

$$F(x_n) + \sigma 2^q \nabla F(x_n) v_n - F(x_n + 2^q v_n) \notin K(F(x_n)),$$

or, equivalently,

$$l(F(x_n), F(x_n) + \sigma 2^q \nabla F(x_n) v_n - F(x_n + 2^q v_n)) \geq 0.$$

Using the fact that l is positively homogeneous, it follows

$$l\left(F(x_n), \frac{F(x_n) - F(x_n + 2^q v_n)}{2^q} + \sigma \nabla F(x_n) v_n\right) \geq 0.$$

Taking the limits for $q \to +\infty$ and $n \to +\infty$, we get

$$l(F(x^*), (\sigma - 1) \nabla F(x^*) v^*) \geq 0$$

and, again by the positive homogeneity of l

$$l(F(x^*), -\nabla F(x^*) v^*) \geq 0. \tag{5.7}$$

On the other hand, by Proposition 5.2, either

$$v_n = 0$$

or
$$l(F(x_n), -\nabla F(x_n)v_n) < 0.$$
So, it holds that
$$l(F(x^*), -\nabla F(x^*)v^*) \leq 0.$$
Combining it with (5.7), the following equality is satisfied:
$$l(F(x^*), \nabla F(x^*)v) = 0$$
and, by Lemma 5.4, this implies that x^* is a weakly stationary point.

Case 2: Finally, we suppose that $\{2^{j_n}\}$ is unbounded. We fix $\varepsilon > 0$ and consider the relation given in (5.4) for $\alpha > 0$ such that $\{2^{j_n} - \alpha\}$ is bounded and $2^{j_n} - \alpha > \varepsilon$. Then we have:
$$F(x_n) + (\sigma 2^{j_n} - \alpha)\nabla F(x_n)v_n - F(x_{n+1}) \in K(F(x_n)).$$
Taking the limit for $n \to +\infty$, and following the ideas used in the first case, we arrive at
$$(2^{j_n} - \alpha)\nabla F(x^*)v^* = 0.$$
Since $(2^{j_n} - \alpha) > \varepsilon > 0$, the result is obtained as in Case (1.a). □

Remark 5.7. In particular, if the domination map $K(\cdot)$ is a Bishop-Phelps-cone of the form $\{z \in Y : \|z\| \leq a(y)z\}$ for all $y \in Y$, the natural representation of $l(y,z)$ is $\|y\| - a(x)y$. Again, if $F \in C^1$, the results are evidently valid.

We end this article with an illustration of Algorithm 2 by a numerical example.

Example 5.8. We consider the problem
$$\min(x+1, x^2+1)^T \text{ s.t. } x \in [0,1]$$
with respect to $K : Y \rightrightarrows Y$,
$$K(y) := \{z : z_1 \geq 0,\ y_2 z_1 - y_1 z_2 \leq 0\}.$$

For finding minimal solutions, first note that for fixed $x^* \in [0,1]$, it is clear that $F(x) = F(x^*)$ if and only if $x^* = x$. So, x^* is a minimizer if and only if x^* is the unique solution of the system
$$(x^*+1, (x^*)^2+1) - (x+1, x^2+1) \in K(F(x^*)).$$

That is
$$x^* - x \geq 0 \tag{5.8}$$
$$((x^*)^2+1)(x^*-x) - (x^*+1)\left[(x^*)^2 - x^2\right] \leq 0. \tag{5.9}$$
Rearranging the last condition, we obtain either $x^* = x$ or
$$(x^*)^2 + 1 - (x^*+1)(x^*+x) \leq 0. \tag{5.10}$$
Note that (5.10) is a linear inequality in x. Since $x^*+1 > x^* \geq 0$, (5.10) is equivalent to proving that the inequality holds for $x = x^*$ (recall (5.8)). Furthermore, as the case $x^* = x$ was already analyzed, by the strict monotonicity of linear functions, the equality can be discarded. So,
$$(x^*)^2 + 1 - 2(x^*+1)x^* < 0 \tag{5.11}$$

must be solved. The zeros of $-(x^*)^2+1-2x^*$ are $-1\pm\sqrt{2}$. Therefore, (5.11) holds for all $x^* \in [0,1] \cap \big((\sqrt{2}-1,+\infty) \cup [-\infty,-1-\sqrt{2})\big) = (\sqrt{2}-1,1)$. This means that $x^* \in (\sqrt{2}-1,1)$ are not minimal points and that for $x^* \in [0,\sqrt{2}-1]$, the unique solution of (5.8)-(5.9) is $x = x^*$. Hence, we can conclude that the set of minimal elements is $[0,\sqrt{2}-1]$. Figure 6 represents this. The values of $F(x)$, $x \in C$, are given by the dashed curve whose solid part corresponds with the minimal points of F with respect to the domination structure given by $K(\cdot)$. The ordering cones for $x = 1$ and $x = 0.75$ shows that the objective function values of these points are not minimal points while at $x = 0.4, x = 0.2, x = 0$, the minimality condition holds.

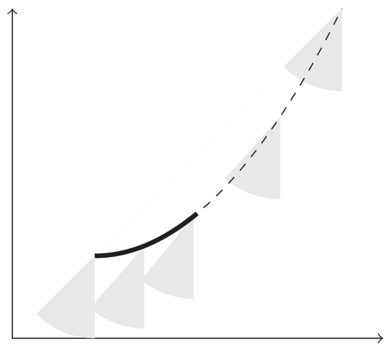

FIGURE 6. $F([0,1])$ and its minimal solutions with respect to $K(\cdot)$

The algorithm is implemented in MatLab R2012 and run on an Intel(R) Atom(TM) CPU N270 at 1.6GHz. Using ten randomly generated starting points in $[0,1]$, the obtained solutions are:

$$x = \begin{array}{c} 0.3152,\ 0.3893,\ 0.3955,\ 0.3822,\ 0.2647, \\ 0.2838,\ 0.4111,\ 0.4136,\ 0.3292,\ 0.3944 \end{array}$$

At most 4 iterations were needed and the largest cputime was .9828 seconds. □

The good behavior of the algorithm in this example provides an idea how this algorithm can be used for other examples.

6. Conclusions

In this paper, we present an extension of a scalarizing functional for vector optimization problems with variable domination structure induced by a set-valued map $K : Y \rightrightarrows Y$. We studied some important properties of the scalarizing functional such as convexity, monotonicity and semi-continuity. Ordering structures given by cone-valued maps were also discussed. The scalarizing functional is useful for a characterization of (weakly) minimal points of vector optimization problems with variable domination structure. It would be very interesting to discuss corresponding results for nondominated elements (see Eichfelder [16]) of these problems.

The properties of the scalarizing functional derived in Section 3 can be used for developing necessary optimality conditions. Furthermore, this scalarization is a possible tool for implementing descent directions algorithms. Future research is focused in these directions.

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