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GENERALIZED SET ORDER RELATIONS AND THEIR NUMERICAL TREATMENT

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Abstract. We introduce very general definitions of set order relations and propose their unified treatment by means of a prominent scalarizing functional. Moreover, we develop a numerical procedure to obtain minimal elements (sets) of a family of finitely many sets. Specifically, several extensions of the well-known Jahn-Graef-Younes method to set optimization are proposed under broad assumptions.

1. Introduction

Many optimization problems are faced with conflicting goals which have to be minimized simultaneously. Such problem structures lead to vector optimization, where different conflicting functions are optimized at the same time. Moreover, most complex multi-objective problems are contaminated with uncertain data. For instance, in traffic optimization, uncertain weather conditions, construction works, or traffic jams can highly influence the computed optimal solutions of a train schedule or shortest path problem. Combining uncertain optimization (especially robust approaches) and vector optimization leads to the field of set optimization, see [2]. Also compare [13] for a recent introduction to set optimization and its applications.

In set optimization, it is important to compare sets by means of set order relations, which are binary relations among sets. There is a variety of set order relations based on convex cones known in the literature (for an overview, see [13, Chapter 2.6.2]).

This paper is concerned with introducing more general set order relations, where the involved set describing the domination structure does not need to be convex. We then characterize these new generalized set order relations by means of a scalarizing functional that is well known from vector optimization. In contrast to [15], the assumptions concerning the domination structure in this paper do not rely on any convexity, and therefore, our results extend those in [15]. Our generalized set order relations are broader than the ones found in the literature, and the assumptions for their representation by means of the scalarizing functional are more general.

The easy structure of the nonlinear scalarizing functional allows for a convenient computation to check whether two sets fulfill the considered new set order relation.

Moreover, we propose a new numerical method for obtaining minimal elements of a family of finitely many sets. Most set optimization problems, even if given in

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a continuous framework, need to be handled in a discrete manner concerning computations. Therefore, given a finite discrete family of sets, in this paper we propose several numerical methods that sort out non-minimal elements and determine all minimal elements of the family of sets. Numerical tests justify that our approaches are useful and the numerical effort is drastically reduced.

2. Preliminaries

Throughout this manuscript, let Y be a linear topological space, and denote the *power set* of Y without the empty set by

$$\mathcal{P}(Y) := \{A \subseteq Y \mid A \text{ is nonempty}\}.$$

We assume that $D \in \mathcal{P}(Y)$ is a closed proper (i.e., $D \neq \{0\}$, and $D \neq Y$) set satisfying the inclusion

$$D + [0, +\infty) \cdot k \subseteq D \tag{2.1}$$

for some $k \in Y \setminus \{0\}$. In $\mathbb{R}^2$, a set D (that is not necessarily a cone) satisfying (2.1) for $k = (1, 1)$ is, for instance, the set $\mathbb{R}^2_+ - \{(0, 1)\}$. If the relation (2.1) is fulfilled, the functional $z^{D,k} \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$z^{D,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - D\} \tag{2.2}$$

is well-defined. We call $z^{D,k}$ *nonlinear scalarizing functional*, as it plays an important role in scalarization methods for obtaining efficient solutions of a vector-valued optimization problem. It can be shown that for a given vector $k \in Y \setminus \{0\}$ and by a variation of the set D satisfying the property (2.1), all efficient elements of a vector optimization problem without any convexity assumptions can be found. The functional $z^{D,k}$ was used to obtain separation theorems for not necessarily convex sets, see [5]. Additionally, numerous applications of $z^{D,k}$ are known in the literature, for instance, coherent risk measures in financial mathematics (see [7]) and uncertain optimization (in particular, in robustness theory, compare [14]). Many properties of $z^{D,k}$ can be found in [5, 6, 23, 24].

Definition 2.1. Let Y be a linear topological space and $\widetilde{D} \in \mathcal{P}(Y)$. A functional $z \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ is **$\widetilde{D}$-monotone** if

$$y_1,\ y_2 \in Y :\ y_1 \in y_2 - \widetilde{D} \Rightarrow z(y_1) \leq z(y_2).$$

Important properties of the functional $z^{D,k}$ which will be used in this paper are given in the following theorem.

Theorem 2.2 ([5,6])**.** *Let Y be a linear topological space, $D \in \mathcal{P}(Y)$ a closed proper set, $\widetilde{D} \in \mathcal{P}(Y)$ and let $k \in Y \setminus \{0\}$ be such that* (2.1) *is satisfied. Then the following properties hold for $z = z^{D,k}$:*

(a) *z is lower semi-continuous.*
(b) (i) *z is convex $\iff$ D is convex,*
 (ii) $[\forall\, y \in Y,\ \forall\, r > 0 :\ z(ry) = rz(y)] \iff D$ *is a cone.*
(c) *z is proper $\iff$ D does not contain lines parallel to k, i.e., $\forall\, y \in Y\ \exists\, r \in \mathbb{R} :\ y + rk \notin D$.*
(d) *z is $\widetilde{D}$-monotone $\iff$ $D + \widetilde{D} \subset D$.*

(e) *z is subadditive* $\Longleftrightarrow D + D \subset D$.

(f) $\forall\ y \in Y,\ \forall\ r \in \mathbb{R}:\ z(y) \le r \Longleftrightarrow y \in rk - D$.

(g) $\forall\ y \in Y,\ \forall\ r \in \mathbb{R}:\ z(y + rk) = z(y) + r$.

(h) *z is finite-valued* $\Longleftrightarrow$ *D does not contain lines parallel to k and* $\mathbb{R}k - D = Y$.

(i) *Let furthermore* $D + (0, +\infty) \cdot k \subset \operatorname{int} D$. *Then z is continuous.*

The following examples illustrate the choice concerning the set D and the vector k in the formulation of the functional $z^{D,k}$.

Example 2.3. (a) Pascoletti, Serafini [21] use the functional $z^{D,k}$ in the special case $Y = \mathbb{R}^n$. Given a function $f : \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^m$, a closed convex cone $D \subset \mathbb{R}^n$ with nonempty interior, parameters $a \in \mathbb{R}^n$, $r \in \operatorname{int} D$, they propose the problem

$$\begin{aligned} &\min t \\ &\text{s.t. } x \in \Omega \\ &\qquad f(x) \in a + tr - D \\ &\qquad t \in \mathbb{R}. \end{aligned}$$

(b) Many well known concepts of proper efficiency (compare [13, Chapter 2.4]) also fit into the general approach of the nonlinear scalarizing concept with the functional $z^{D,k}$. Since many of them are based on a certain kind of generalized linear scalarization, they are endowed with a polyhedral structure: In [25], Weidner characterizes properly efficient elements in the sense of Geoffrion by solutions of the auxiliary problem

$$\min_{y \in \mathbb{R}^n} \max_{i=1,\dots,n} (\langle v_i, y\rangle - \nu_i)$$

with $v_i \in \operatorname{int} \mathbb{R}^n_+$, $\sum_{j=1}^n v_i^j = 1$, $\nu_i \in \mathbb{R}$, $i = 1, \dots, n$. Without effort, we can verify that these auxiliary problems coincide with the problem $\min_{y \in \mathbb{R}^n} z^{D,k}$ for $D := \{y \in \mathbb{R}^n\ :\ \forall\ i = 1, \dots, n : \langle v_i, y\rangle - \nu_i \ge 0\}$ and $k := (1, \dots, 1)^T \in \mathbb{R}^n$.

(c) Kaliszewski [12] characterizes efficiency in vector optimization with respect to polyhedral cones by some inconsistency assertions. He uses a polyhedral cone D given by

$$D := \{y \in \mathbb{R}^n :\ \langle -b_i, y\rangle \ge 0,\ i = 1, ..., m\}$$

with $b_i \in \mathbb{R}^n$, $i = 1, \dots, m$. The inconsistency notions he uses can equivalently be represented by means of the functional $z^{D,k}$, as was shown by Tammer and Winkler in [22].

3. Generalized set order relations

In this section, we introduce generalized set order relations in order to formulate the solution concepts in Section 4. Our intention is to study set-valued optimization problems with general set order relations and to derive corresponding algorithms. In the following definition, we introduce a generalized set order relation w.r.t. a nonempty subset D of Y. The following set order relation generalizes the upper set less order relation by Kuroiwa [17, 18], where the involved set D is a convex cone.

Definition 3.1 (Generalized Upper Set Less Order Relation, [16])**.** Let $D \in \mathcal{P}(Y)$. The **generalized upper set less order relation** $\preceq_D^u$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq_D^u B :\Longleftrightarrow A \subseteq B - D,$$

which is equivalent to

$$\forall\, a \in A,\ \exists\, b \in B:\ a \in b - D.$$

Remark 3.2. Notice that $\preceq_D^u$ is transitive if $D + D \subseteq D$. If D is a cone, then $D+D \subseteq D$ implies that D is convex. If, for instance, $D = \mathbb{R}^2_+ \setminus \{0\}$, then $D+D \subseteq D$ is fulfilled, but D is not a cone. Moreover, $\preceq_D^u$ is reflexive if $0 \in D$. Therefore, $\preceq_D^u$ is a preorder if $D + D \subseteq D$ and $0 \in D$.

The following result has been shown in [16].

Theorem 3.3. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, $k \in Y \setminus \{0\}$ such that* (2.1) *is fulfilled, $\widetilde{D} \subseteq Y$ such that $D + \widetilde{D} \subseteq D$, and $A, B \in \mathcal{P}(Y)$. Then it holds*

$$A \subseteq B - \widetilde{D} \implies \sup_{a\in A} z^{D,k}(a) \le \sup_{b\in B} z^{D,k}(b).$$

The following result, shown in [16], gives an equivalent representation for $A \preceq_D^u B$.

Theorem 3.4. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, the following implication holds:*

$$A \subseteq B - D \quad \Longrightarrow \quad \sup_{a\in A}\ \inf_{b\in B} z^{D,k}(a-b) \le 0\,. \tag{3.1}$$

Assume on the other hand, that there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{b\in B} z^{D,k_0}(a - b)$ is attained for all $a \in A$, then the converse is also true, i.e.,*

$$\sup_{a\in A}\ \inf_{b\in B} z^{D,k_0}(a-b) \le 0 \quad \Longrightarrow \quad A \subseteq B - D\,. \tag{3.2}$$

In this paper, our goal is to study different extensions of several known set order relations and their representation by means of the functional $z^{D,k}$. We start by introducing the following extension of the lower set less order relation by Kuroiwa [17, 18].

Definition 3.5 (Generalized Lower Set Less Order Relation)**.** Let $D \in \mathcal{P}(Y)$. The **generalized lower set less order relation** $\preceq_D^l$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq_D^l B :\Longleftrightarrow B \subseteq A + D,$$

which is equivalent to

$$\forall\, b \in B,\ \exists\, a \in A:\ b \in a + D.$$

Remark 3.6. Notice that $\preceq_D^l$ is transitive if $D+D \subseteq D$ and it is reflexive if $0 \in D$.

If the set D in Definition 3.5 is replaced by a convex cone $C \subset Y$, then this definition coincides with the definition of the lower set less order relation introduced by Kuroiwa [17, 18], and $B \subseteq A + C$ can be replaced by

$$\forall\, b \in B,\ \exists\, a \in A:\ a \le_C b\,,$$

where $\leq_C$ relates to the order relation induced by the convex cone C, thus, $a \leq_C b$ means that $a \in b - C$.

The following theorem gives a first insight into the relationships between the generalized lower set less order relation and the functional $z^{D,k}$.

Theorem 3.7. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, $k \in Y \setminus \{0\}$ such that* (2.1) *is fulfilled, let $\widetilde{D} \subseteq Y$ such that $D + \widetilde{D} \subseteq D$, and $A, B \in \mathcal{P}(Y)$. Then it holds*

$$B \subseteq A + \widetilde{D} \implies \inf_{a \in A} z^{D,k}(a) \leq \inf_{b \in B} z^{D,k}(b).$$

Proof. Choose an arbitrary vector $k \in Y \setminus \{0\}$ such that (2.1) is satisfied, and let $B \subseteq A + \widetilde{D}$. Then, we have

$$\forall\, b \in B,\ \exists\, a \in A:\ b \in a + \widetilde{D}\,.$$

The monotonicity property of the functional $z^{D,k}$ (compare Theorem 2.2 (d)) yields

$$\forall\, b \in B,\ \exists\, a \in A:\ z^{D,k}(a) \leq z^{D,k}(b)\,.$$

Therefore, we conclude with the stated inequality. □

The following corollary, which was proven in [15, Theorem 3.15], is a consequence of Theorem 3.7.

Corollary 3.8. *Let $D \in \mathcal{P}(Y)$ be a closed proper convex cone in Y, $k \in Y \setminus \{0\}$, $A, B \in \mathcal{P}(Y)$. Then it holds*

$$B \subseteq A + D \implies \inf_{a \in A} z^{D,k}(a) \leq \inf_{b \in B} z^{D,k}(b).$$

We derive the following result in correspondence with Theorem 3.4 for the generalized lower set less order relation.

Theorem 3.9. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, the following implication holds:*

$$B \subseteq A + D \quad \implies \quad \sup_{b \in B}\ \inf_{a \in A} z^{D,k}(a - b) \leq 0\,.$$

On the other hand, assume that there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{a \in A} z^{D,k_0}(a - b)$ is attained for all $b \in B$, then*

$$\sup_{b \in B}\ \inf_{a \in A} z^{D,k_0}(a - b) \leq 0 \quad \implies \quad B \subseteq A + D\,.$$

Proof. Let $B \subseteq A + D$. This means

$$\forall\, b \in B,\ \exists\, a \in A:\ b \in a + D \quad \implies \quad \forall\, b \in B,\ \exists\, a \in A:\ a - b \in -D\,.$$

Because of Theorem 2.2 (f) with $r = 0$ and $y = a - b$, we have

$$\forall\, b \in B,\ \exists\, a \in A:\ z^{D,k}(a - b) \leq 0\,,$$

and this implies

$$\sup_{b \in B}\ \inf_{a \in A} z^{D,k}(a - b) \leq 0\,.$$

Conversely, let $k_0 \in Y \setminus \{0\}$ be given such that for all $b \in B$ the infimum $\inf_{a\in A} z^{D,k_0}(a-b)$ is attained. Let

$$\sup_{b\in B} \inf_{a\in A} z^{D,k_0}(a-b) \leq 0\,. \tag{3.3}$$

That means

$$\forall\ b \in B:\ \inf_{a\in A} z^{D,k_0}(a-b) \leq 0\,.$$

Because for all $b \in B$ the infimum $\inf_{a\in A} z^{D,k_0}(a-b)$ is attained, we obtain

$$\forall\ b \in B\ \exists\ \overline{a} \in A:\ z^{D,k_0}(\overline{a}-b) = \inf_{a\in A} z^{D,k_0}(a-b) \leq 0\,.$$

By Theorem 2.2 (f) with $r = 0$ and $y = a - b$, we conclude with

$$\forall\ b \in B\ \exists\ \overline{a} \in A:\ \overline{a} - b \in -D,$$

thus $B \subseteq A + D$. $\square$

In the following definition, we extend the notion of the set less order relation (see Young [27] and Nishnianidze [20]).

Definition 3.10 (Generalized Set Less Order Relation)**.** Let $D \in \mathcal{P}(Y)$. The **generalized set less order relation** $\preceq^s_D$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq^s_D B :\Longleftrightarrow A \preceq^u_D B \text{ and } A \preceq^l_D B.$$

The next result follows directly from Theorems 3.4 and 3.9.

Corollary 3.11. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, we have*

$$A \preceq^s_D B \quad \Longrightarrow \quad \sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b) \leq 0 \text{ and } \sup_{b\in B} \inf_{a\in A} z^{D,k}(a-b) \leq 0.$$

If, on the other hand, there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{b\in B} z^{D,k_0}(a-b)$ is attained for all $a \in A$, and if there exists $k_1 \in Y \setminus \{0\}$ satisfying* (2.1) *such that $\inf_{a\in A} z^{D,k_1}(a-b)$ is attained for all $b \in B$, then*

$$A \preceq^s_D B \quad \Longleftarrow \quad \sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a-b) \leq 0 \text{ and } \sup_{b\in B} \inf_{a\in A} z^{D,k_1}(a-b) \leq 0.$$

The following definition is an extension of the certainly less order relation (see Jahn, Ha [10], Eichfelder, Jahn [4]).

Definition 3.12 (Generalized Certainly Less Order Relation)**.** Let $D \in \mathcal{P}(Y)$. The **generalized certainly less order relation** $\preceq^c_D$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq^c_D B :\Longleftrightarrow (A = B) \text{ or } (\forall\ a \in A,\ \forall\ b \in B:\ a \in b - D)\,.$$

The following result does not require any attainment property. We omit its proof, as it is similar to that of Theorem 3.9.

Theorem 3.13. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, the following equivalence holds:*

$$\forall\ a \in A,\ \forall\ b \in B:\ a \in b - D \quad \Longleftrightarrow \quad \sup_{(a,b)\in A\times B} z^{D,k}(a-b) \leq 0\,.$$

Corollary 3.14. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, $k \in Y \setminus \{0\}$ such that* (2.1) *is fulfilled, $A, B \in \mathcal{P}(Y)$. Then we have the following equivalence for the generalized certainly less order relation:*

$$A \preceq_D^c B \quad \Longleftrightarrow \quad (A = B) \;\; or \; \left(\sup_{(a,b)\in A\times B} z^{D,k}(a-b) \leq 0 \right).$$

Notice that the result in Corollary 3.14 holds true for arbitrary $k \in Y \setminus \{0\}$ fulfilling (2.1). So we can conclude that $A \preceq_D^c B$ is equivalent to

$$(A = B) \;\text{ or } \left(\forall\, k \in Y \setminus \{0\} \text{ satisfying (2.1):} \sup_{(a,b)\in A\times B} z^{D,k}(a-b) \leq 0 \right).$$

The next definition is a more general form of the possibly less order relation (see [1, 10]).

Definition 3.15 (Generalized Possibly Less Order Relation)**.** Let $D \in \mathcal{P}(Y)$. The **generalized possibly less order relation** $\preceq_D^p$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq_D^p B :\Longleftrightarrow \exists\, a \in A,\; \exists\, b \in B: \; a \in b - D.$$

The following result shows that the nonlinear scalarizing functional $z^{D,k}$ is useful for the characterization of the generalized possibly less order relation.

Theorem 3.16. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, the following implication holds:*

$$\exists\, a \in A,\; \exists\, b \in B: \; a \in b - D \quad \Longrightarrow \quad \inf_{(a,b)\in A\times B} z^{D,k}(a-b) \leq 0..$$

If there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{(a,b)\in A\times B} z^{D,k_0}(a-b)$ is attained, we have:*

$$\inf_{(a,b)\in A\times B} z^{D,k_0}(a-b) \leq 0 \quad \Longrightarrow \quad \exists\, a \in A, \exists\, b \in B: \, a \in b - D\,.$$

Remark 3.17. Of course, many other set order relations can be found in the literature. Some of them can be generalized in the way we conducted so far. For example, the minmax less order relation and the minmax certainly less order relation, given in Jahn, Ha [10] can be generalized and expressed via the nonlinear scalarizing functional $z^{D,k}$. Moreover, in Kuroiwa et al. [19] the following order relations are presented (with D being a proper closed convex cone):

$$A \preceq^{(\text{ii})} B \iff \exists\, a \in A: \; \forall\, b \in B, \; a \in b - D$$

and

$$A \preceq^{(\text{iv})} B \iff \exists\, b \in B: \; \forall\, a \in A, \; a \in b - D.$$

Under appropriate attainment properties and if D and $k \in Y \setminus \{0\}$ satisfy (2.1), these relations are concerned with

$$\inf_{a\in A} \sup_{b\in B} z^{D,k}(a-b) \leq 0 \quad \text{and} \quad \inf_{b\in B} \sup_{a\in A} z^{D,k}(a-b) \leq 0.$$

However, we will not pursue them any further, as they are similar to $\preceq_D^u$ as well as $\preceq_D^l$, and coincide by simply interchanging the infima and suprema.

4. Numerical methods for determining minimal elements

This section is concerned with developing numerical methods for finding minimal elements of a family of sets with respect to the new generalized set order relations that we introduced in Section 3.

In the literature, there already exist some algorithms for solving set-valued optimization problems. Jahn [9] proposes a descent method that generates approximations of minimal elements of set-valued optimization problems under convexity assumptions on the considered sets. In [9], the set less order relation is characterized by means of linear functionals. More recently, in [15], the authors propose a similar descent method for obtaining approximations of minimal elements of set-valued optimization problems. In [15], several set order relations are characterized by the nonlinear scalarizing functional $z^{D,k}$, where D is assumed to be a proper convex cone. Since the nonlinear functional $z^{D,k}$ is used in [15], no convexity assumptions are needed. The approaches in [9,15] all rely on set order relations where the involved domination structure is given by cones.

Our approach in this paper is two-fold: First, we extend the well-known Jahn-Graef-Younes method, which was introduced in [26] for vector optimization problems. The Jahn-Graef-Younes method selects minimal elements of a set of **finitely** many elements. Its advantage is that this method reduces the numerical effort by excluding elements which cannot be minimal for a given set. In this section, we extend this method to the set-valued case in order to obtain minimal elements of a family of **finitely** many sets. We propose several extensions of the Jahn-Graef-Younes method under different assumptions on the generalized set order relations introduced in Section 3. Secondly, when the involved sets are compared by means of any of those proposed set order relations, we use the results from Section 3 to evaluate $A \preceq B$ by using the nonlinear scalarizing functional $z^{D,k}$.

First, we recall the definition of minimal elements.

Definition 4.1 (Minimal Elements of a Family of Sets)**.** Let $\mathcal{A}$ be a family of nonempty subsets of Y and let a set order relation $\preceq$ on $\mathcal{P}(Y)$ be given. $\overline{A} \in \mathcal{A}$ is called a **minimal element** of $\mathcal{A}$ w.r.t. $\preceq$ if

$$A \preceq \overline{A},\ A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq A\,.$$

The set of all minimal elements of $\mathcal{A}$ w.r.t. $\preceq$ is denoted by $\mathcal{A}_{\preceq}$.

When the family of sets $\mathcal{A}$ is given by a large number of elements, it may take a long time to compare the sets pairwise according to Definition 4.1. We propose a method that significantly reduces the number of comparisons of sets. Reducing the numerical effort is especially useful if each comparison is rather expensive. The following algorithm filters out elements of a family of sets which cannot be minimal. This procedure extends the Jahn-Graef-Younes method which is given in the dissertation by Younes [26], Jahn and Rathje [11] (compare also Jahn [8, Section 12.4]) for minimal elements in the vector-valued case, where $Y = \mathbb{R}^n$. Eichfelder [3] formulated corresponding algorithms for vector-valued problems with a variable ordering structure. We extend the idea of such a method to set optimization problems, where we assume that a family of finitely many sets $\mathcal{A}$ is given and minimal elements of $\mathcal{A}$ are to be identified.

Algorithm 4.2. (Jahn-Graef-Younes method for sorting out non-minimal elements of a family of finitely many sets)

Input: $\mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{R}^n$, set order relation $\preceq$
% initialization
$\mathcal{T} := \{A_1\}$,
% iteration loop
for $j = 2 : 1 : m$ **do**
  if $(A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
  end if
end for
Output: $\mathcal{T}$

Algorithm 4.2 is a reduction method which sorts out sets that cannot be minimal. In the if-statement of Algorithm 4.2, each element is compared only with elements that have been considered so far (which belong to the set $\mathcal{T}$), so it is not necessary to compare all elements with each other pairwise, which can reduce the computation time of determining minimal elements significantly. Notice that the conditions $A \preceq A_j$ and $A_j \preceq A$ in the if-statement in Algorithm 4.2 can be evaluated by means of computing the nonlinear scalarizing functional $z^{D,k}$ (compare Theorems 3.4, 3.9, 3.16 and Corollaries 3.11 and 3.14 for representations of different order relations by means of $z^{D,k}$). This will be done on page 55. Below we show that all minimal elements of the family of sets $\mathcal{A}$ are contained in the output set $\mathcal{T}$ generated by Algorithm 4.2.

Theorem 4.3. (1) *Algorithm 4.2 is well-defined.*
(2) *Algorithm 4.2 generates a nonempty set $\mathcal{T} \subseteq \mathcal{A}$.*
(3) *Every minimal element of $\mathcal{A}$ also belongs to the set $\mathcal{T}$ generated by Algorithm 4.2.*

Proof. As 1. and 2. are obvious, we only prove part 3. Let A_j be a minimal element of $\mathcal{A}$, but assume that $A_j \notin \mathcal{T}$. Clearly $j \neq 1$. As A_j is a minimal element of $\mathcal{A}$, we have

$$A \preceq A_j,\ A \in \mathcal{A} \implies A_j \preceq A.$$

Since $\mathcal{T} \subseteq \mathcal{A}$, we have

$$A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A.$$

But then the condition in the if-statement is fulfilled and A_j is added to $\mathcal{T}$, which is a contradiction to our assumption. □

As mentioned before, the conditions $A \preceq A_j$ and $A_j \preceq A$ in the if-statement in Algorithm 4.2 shall be evaluated by means of the nonlinear scalarizing functional $z^{D,k}$ for all introduced set order relations. This will be done on page 55. First, we consider the following attainment properties:

Assumption 4.4 (Attainment Property).

(u) Assume that there exist k_0^u, $k_1^u \in Y \setminus \{0\}$ satisfying (2.1) such that $\inf_{\overline{a}\in A_j} z^{D,k_0^u}(a-\overline{a})$ is attained for all $a \in A$ and $\inf_{a\in A} z^{D,k_1^u}(\overline{a}-a)$ is attained for all $\overline{a} \in A_j$.

(l) Assume that there exist k_0^l, $k_1^l \in Y \setminus \{0\}$ satisfying (2.1) such that $\inf_{a\in A} z^{D,k_0^l}(a-\overline{a})$ is attained for all $\overline{a} \in A_j$ and $\inf_{\overline{a}\in A_j} z^{D,k_1^l}(\overline{a}-a)$ is attained for all $a \in A$.

(s) Assume that there exist k_0^s, k_1^s, k_2^s, $k_3^s \in Y \setminus \{0\}$ satisfying (2.1) such that $\inf_{\overline{a}\in A_j} z^{D,k_0^s}(a-\overline{a})$ is attained for all $a \in A$, $\inf_{a\in A} z^{D,k_1^s}(\overline{a}-a)$ is attained for all $\overline{a} \in A_j$, $\inf_{a\in A} z^{D,k_2^s}(a-\overline{a})$ is attained for all $\overline{a} \in A_j$ and $\inf_{\overline{a}\in A_j} z^{D,k_3^s}(\overline{a}-a)$ is attained for all $a \in A$.

(p) Assume that there exist $k_0^p, k_1^p \in Y \setminus \{0\}$ satisfying (2.1) such that $\inf_{(a,\overline{a})\in A\times A_j} z^{D,k_0^p}(a-\overline{a})$ and $\inf_{(a,\overline{a})\in A\times A_j} z^{D,k_1^p}(\overline{a}-a)$ are attained.

Remark 4.5. The attainment properties above are important for the representation of the introduced generalized set order relations by means of the nonlinear scalarizing functional $z^{D,k}$ (compare Theorems 3.4, 3.9, 3.16 and Corollary 3.11). Sufficient conditions ensuring the existence of solutions of corresponding optimization problems (extremal principles) are given in the literature. The well-known Theorem of Weierstrass says that a lower semi-continuous function on a nonempty compact set has a minimum. An extension of the Theorem of Weierstrass is given by Zeidler [28, Proposition 9.13]: A proper lower semi-continuous and quasi-convex function on a nonempty closed bounded convex subset of a reflexive Banach space has a minimum. Taking into account that the functional z^{D,k_0} is lower semi-continuous and convex if $D \subset Y$ is a proper closed convex cone and $k_0 \in D\setminus\{0\}$ (compare Theorem 2.2), we get that the attainment property for $\inf_{a\in A} z^{D,k_0}(a-b)$ (with $b \in B$ fixed) is fulfilled if A is a nonempty closed bounded convex subset of a reflexive Banach space and D is a proper closed convex cone.

In the following, we will give an implementation of the implication $A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A$ in Algorithm 4.2 in order to show how we are using the results from Section 3 for deriving the algorithm. Especially in Step 5 of the following implementation of Algorithm 4.2 it can be seen that the results concerning the scalarizing functional $z^{D,k}$ are important for computing minimal elements of the set $\mathcal{A}$. In the following, we assume that the set order relation used in Algorithm 4.2 is given by $\preceq_D^t$, where t is replaced by u, l, s, c, p for the generalized *upper set less order relation* $\preceq_D^u$, *lower set less order relation* $\preceq_D^l$, *set less order relation* $\preceq_D^s$, *certainly set less order relation* $\preceq_D^c$ or *possibly set less order relation* $\preceq_D^p$, respectively.

In the **if**-statement of Algorithm 4.2, the implication $A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A$ is evaluated. Here, it is our goal to check this implication by means of the nonlinear scalarizing functional $z^{D,k}$ and the results from Section 3 for $\preceq = \preceq_D^t$.

We use the following implications (see Theorems 3.4, 3.9, 3.16 and Corollaries 3.11 and 3.14) in our implementation of Algorithm 4.2 (note that these are equivalent to $A \preceq A_j \implies A_j \preceq A$ under appropriate attainment properties):

$$(I_u) \qquad \sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_0^u}(a-\overline{a}) \le 0 \quad \Longrightarrow \quad \sup_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_1^u}(\overline{a}-a) \le 0$$

$$(I_l) \qquad \sup_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_0^l}(a-\overline{a}) \le 0 \quad \Longrightarrow \quad \sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_1^l}(\overline{a}-a) \le 0$$

$$(I_s) \qquad \begin{cases} \quad \sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_0^s}(a-\overline{a}) \le 0 \wedge \sup_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_2^s}(a-\overline{a}) \le 0 \\ \Longrightarrow \sup_{\overline{a}\in \overline{A}}\inf_{a\in A} z^{D,k_1^s}(\overline{a}-a) \le 0 \wedge \sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_3^s}(\overline{a}-a) \le 0 \end{cases}$$

$$(I_c) \qquad \begin{cases} \qquad (A = A_j) \ \vee \ \sup_{a\in A}\sup_{\overline{a}\in A_j} z^{D,k}(a-\overline{a}) \le 0 \\ \Longrightarrow (A = A_j) \ \vee \ \sup_{\overline{a}\in A_j}\sup_{a\in A} z^{D,k}(\overline{a}-a) \le 0 \end{cases}$$

$$(I_p) \qquad \inf_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_0^p}(a-\overline{a}) \le 0 \quad \Longrightarrow \quad \inf_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_1^p}(\overline{a}-a) \le 0$$

The following implementation of Algorithm 4.2 checks whether the implication $A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A$ in the **if**-statement in Algorithm 4.2 is fulfilled for some input A_j, given $\mathcal{T}$, and $t \in \{u,l,s,c,p\}$ for $\preceq_D^t := \preceq$ which was chosen in the input of Algorithm 4.2. Note that the set D and $t \in \{u,l,s,c,p\}$ were already chosen in the input of Algorithm 4.2. If this implication is satisfied for all $A \in \mathcal{T}$, then the set A_j is added to the family of sets $\mathcal{T}$. Then the **for**-loop in Algorithm 4.2 continues with $j := j+1$. If this implication is not fulfilled for some $A \in \mathcal{T}$, then the **for**-loop in Algorithm 4.2 continues with $j := j+1$, but the set A_j is not added to the family of sets $\mathcal{T}$. Note that the set $K := \{k \in Y \setminus \{0\} \mid D + [0,+\infty)\cdot k \subseteq D\}$, which is necessary for the definition of the functional $z^{D,k}$, as $k \in K$, should be determined at the beginning of Algorithm 4.2.

Realization the implication $A \preceq_D^t A_j,\ A \in \mathcal{T} \implies A_j \preceq_D^t A$ in Algorithm 4.2:

Input: $\mathcal{T}$ and j

Step 1: Set $\widetilde{\mathcal{T}} := \mathcal{T}$. Go to Step 2.

Step 2: If $\widetilde{\mathcal{T}} = \emptyset$, then the implication $A \preceq_D^t A_j,\ A \in \mathcal{T} \implies A_j \preceq_D^t A$ holds and STOP. Otherwise, go to Step 3.

Step 3: Choose $A \in \widetilde{\mathcal{T}}$. Set $\widetilde{\mathcal{T}} := \widetilde{\mathcal{T}} \setminus \{A\}$. Go to Step 4.

Step 4: When $t \in \{u,l,p,s\}$, choose $k_r^t \in K$ ($r = 0,1$ if $t \in \{u,l,p\}$, $r = 0,1,2,3$ if $t = s$) such that Assumption 4.3 (t) is fulfilled. When $t = c$, choose $k \in K$. Go to Step 5.

Step 5: If the implication (I_t) is true, then go to Step 2. Otherwise, the implication does not hold and STOP.

Remark 4.6. The above implementation of the implication $A \preceq_D^t A_j,\ A \in \mathcal{T} \implies A_j \preceq_D^t A$ Algorithm 4.2 is especially easy for the generalized certainly less order relation $\preceq_D^c$ (when $t = c$), as no attainment property needs to be fulfilled for this particular set order relation (compare Theorem 3.13).

Example 4.7. Let $D := \mathbb{R}^2_+$ and $\preceq := \preceq_D^c$. We have randomly computed 1,000 sets, for easy comparison each set is a ball of radius one in $\mathbb{R}^2$. Out of those 1,000 sets, a total number of 93 are minimal w.r.t. to $\preceq$. Algorithm 4.2 generates 103 sets in $\mathcal{T}$, which is already a reduction of 897 sets. In Figure 1 the elements of the set $\mathcal{T}$ are the filled circles.

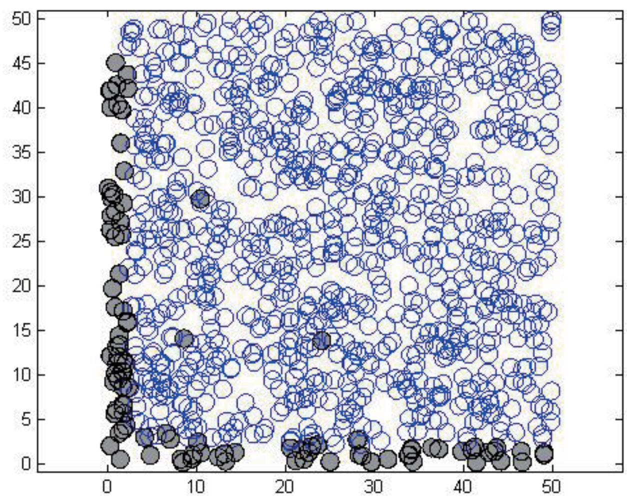

FIGURE 1. A randomly generated family of sets. The filled circles belong to the set $\mathcal{T}$ generated by Algorithm 4.2.

Remark 4.8. Notice that the set order relation $\preceq$ does not need to be transitive in Algorithm 4.2, in contrast to descent methods (see Jahn [9]), which rely on the transitivity of the considered set order relation.

Example 4.9. Let $D := \mathbb{R}^2_+$, $\preceq := \preceq_D^c$, $A_1 := B_1(3,3)$, $A_2 := B_1(5,5)$, $A_3 := B_1(0,0)$ (where $B_1(y_1, y_2)$ denotes the closed ball of radius one around the point $(y_1, y_2) \in \mathbb{R}^2$). Let the family of sets be given by these balls, i.e., $\mathcal{A} := \{A_1, A_2, A_3\}$. The only minimal element of $\mathcal{A}$ w.r.t. $\preceq$ is $A_3 = B_1(0,0)$. Algorithm 4.2 generates the set $\mathcal{T} := \{A_1, A_3\}$.

Applying the for-loop in Algorithm 4.2 backwards leads to the following algorithm, which determines all minimal elements of a family of sets under an external stability assumption on the set of minimal elements $\mathcal{A}_{\preceq}$, when the set order relation is antisymmetric. For example, the generalized certainly less order relation $\preceq_D^c$ is antisymmetric if D is a **pointed** cone (see Proposition 4.18).

Definition 4.10. If for all non-minimal elements $A \in \mathcal{A} \backslash \mathcal{A}_{\preceq}$ there exists a minimal element $\overline{A} \in \mathcal{A}_{\preceq}$ with $\overline{A} \preceq A$, then $\mathcal{A}_{\preceq}$ is called **externally stable**.

Algorithm 4.11. (Jahn-Graef-Younes method with backward iteration for finding minimal elements of a family of finitely many sets, where $\mathcal{A}_{\preceq}$ is externally stable)

Input: $\mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{R}^n$, antisymmetric set order relation $\preceq$
% initialization
$\mathcal{T} := \{A_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ **do**
  if $(A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
  end if
end for
$\{A_1, \ldots, A_p\} := \mathcal{T}$
$\mathcal{U} := \{A_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ **do**
  if $(A \preceq A_j,\ A \in \mathcal{U} \implies A_j \preceq A)$ **then**
    $\mathcal{U} := \mathcal{U} \cup \{A_j\}$
  end if
end for
Output: $\mathcal{U}$

Theorem 4.12. *Let the set order relation $\preceq$ be antisymmetric and the set of minimal elements $\mathcal{A}_{\preceq}$ be nonempty and externally stable. Then the output $\mathcal{U}$ of Algorithm 4.11 consists of exactly all minimal elements of the family of sets $\mathcal{A}$.*

Proof. Let $\mathcal{U} := \{A_1, \ldots, A_q\}$. By 3 of Theorem 4.3, we know that all minimal elements of $\mathcal{A}$ are contained in $\mathcal{T}$ as well as in $\mathcal{U}$. Now we prove that every element of $\mathcal{U}$ is also a minimal element of the set $\mathcal{A}$. Let $A_j \in \mathcal{U}$ be arbitrarily chosen. By the forward iteration of Algorithm 4.11, we obtain

$$\forall\ i < j\ (i \geq 1):\ A_i \preceq A_j \Longrightarrow A_j \preceq A_i.$$

The backward iteration of Algorithm 4.11 yields

$$\forall\ i > j\ (i \leq q):\ A_i \preceq A_j \Longrightarrow A_j \preceq A_i.$$

This means that

$$\forall\ i \neq j\ (1 \leq i \leq q):\ A_i \preceq A_j \Longrightarrow A_j \preceq A_i. \tag{4.1}$$

(4.1) implies that

$$\forall\ A_i \in \mathcal{U} \setminus \{A_j\}:\ A_i \preceq A_j \quad \Longrightarrow \quad A_j \preceq A_i.$$

Then, A_j is a minimal element of $\mathcal{U}$. Now suppose that A_j is not a minimal element in $\mathcal{A}$, then $A_j \notin \mathcal{A}_{\preceq}$. Then, as $\mathcal{A}_{\preceq}$ was assumed to be externally stable, there exists a minimal element A in $\mathcal{A}_{\preceq}$ (especially, $A \neq A_j$) with the property $A \preceq A_j$. Since A is a minimal element in $\mathcal{A}$, Theorem 4.3, 3. implies that $A \in \mathcal{U}$. Therefore, by (4.1), $A_j \preceq A$, as A_j is minimal in $\mathcal{U}$ and $A \in \mathcal{U}$. By the antisymmetry of the set order relation $\preceq$, we obtain $A = A_j$, a contradiction. $\square$

It is again possible to formulate an implementation of the implication $A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A$ of Algorithm 4.11. This can be performed for the first for-loop analogously to the process on page 55, and for the second for-loop simply by replacing $\mathcal{T}$ by $\mathcal{U}$ and changing $j := j + 1$ to $j := j - 1$.

Example 4.13. We return to Example 4.9. The backward iteration in Algorithm 4.11 generates the set $\mathcal{U} = \{A_3\}$, which is exactly the minimal element of $\mathcal{A}$ w.r.t. $\preceq$.

Example 4.14. The minimal elements of the randomly generated family of sets of Example 4.7 are illustrated as dark filled circles in Figure 2. The remaining elements which are lighter belong to the set $\mathcal{T}$, but not to $\mathcal{U}$.

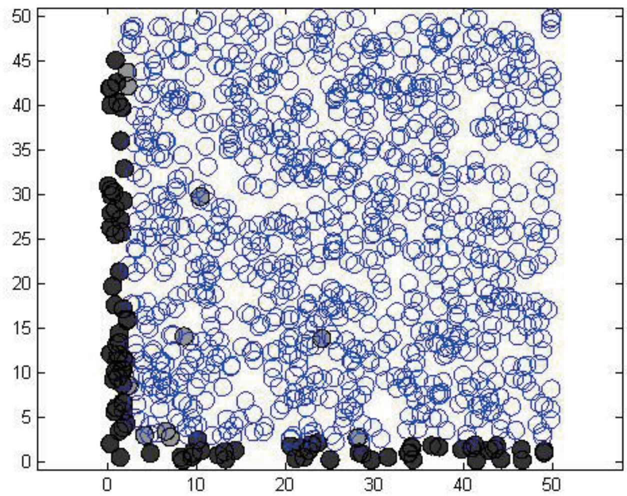

FIGURE 2. A randomly generated family of sets. The minimal elements w.r.t. $\preceq_D^c$ are dark, the lighter sets belong to the set $\mathcal{T}$ generated by Algorithm 4.11.

In the following, we give a sufficient condition for the set of minimal elements $\mathcal{A}_{\preceq}$ to be externally stable.

Lemma 4.15. *Let a family $\mathcal{A}$ of finitely many nonempty subsets of Y be given and let the set order relation $\preceq$ be transitive and antisymmetric. Assume that the set of minimal elements w.r.t. $\preceq$, denoted as $\mathcal{A}_{\preceq}$, is nonempty. Then $\mathcal{A}_{\preceq}$ is externally stable.*

Proof. Let some $A \in \mathcal{A}$, and A is assumed to be not minimal w.r.t. $\preceq$. Then there exists some $A_1 \in \mathcal{A}$ such that $A_1 \preceq A$ and $A \not\preceq A_1$. If $A_1 \in \mathcal{A}_{\preceq}$, then there is nothing to show. If $A_1 \notin \mathcal{A}_{\preceq}$, then there exists some $A_2 \in \mathcal{A}$ with $A_2 \preceq A_1$ and $A_1 \not\preceq A_2$. As $\preceq$ is transitive, we get $A_2 \preceq A$. As $\mathcal{A}$ consists of finitely many elements and $\preceq$ is antisymmetric, this procedure stops with a minimal element. $\square$

Remark 4.16. Here we briefly explain the difference between our extension of the Jahn-Graef-Younes-Algorithm to set optimization with the originally introduced version by Younes (compare [8, Section 12.4]). Let $Y = \mathbb{R}^n$ with the ordering $\leq_C$ induced by a closed convex cone $C \subset Y$. The if-statement in the original Jahn-Graef-Younes-Algorithm in vector optimization reads

$$\text{for all } y \in \mathcal{T} \setminus \{\overline{y}\} : \; y \not\leq_C y_j,$$

and transferring this notion to our set optimization setting would yield the condition

$$\text{for all } A \in \mathcal{T} \setminus \{A_j\} : \; A \not\preceq A_j.$$

However, then the set $\mathcal{T}$ generated by Algorithm 4.2 would possibly not contain all minimal elements. The reason for this is the following: We work with the minimality notion given in Definition 4.1:

$$A \preceq \overline{A}, \; A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq A \,. \tag{4.2}$$

However, the implication (4.2) does not imply

$$\forall \; A \in \mathcal{A} \setminus \{\overline{A}\} : \; A \not\preceq \overline{A}, \tag{4.3}$$

unless $\preceq$ is antisymmetric. We note that (4.3) always implies (4.2), even if $\preceq$ is not antisymmetric. We exemplarily illustrate this with a small example in vector optimization. Let $a = (a^1, a^2) \in \mathbb{R}^2$ be given, $C := \{y \in \mathbb{R}^2 \mid a^T y \geq 0\}$, $\mathcal{A} = \{y \in \mathbb{R}^2 \mid a^T y = 0\}$ and $\overline{A} \in \mathcal{A}$ arbitrarily given. The binary relation $\leq_C := \preceq$ is defined as $y_1 \leq_C y_2 \;:\Longleftrightarrow y_1 \in y_2 - C$. Then all elements in $\mathcal{A}$ are minimal w.r.t. $\preceq$. Then (4.2) is satisfied for all $A = y \in \mathcal{A}$. However, we have for all $y_1, y_2 \in \mathcal{A}$ the relation $y_1 \leq_C y_2$. Therefore, (4.3) does not hold true for any $A = y \in \mathcal{A}$. The reason, of course, is that the cone C is a halfspace and therefore not pointed, hence the binary relation $\leq_C$ is not antisymmetric.

Proposition 4.17. *We consider the statements*

$$\nexists \; A \in \mathcal{A} \setminus \{\overline{A}\} : \; A \preceq \overline{A} \tag{4.4}$$

and

$$A \preceq \overline{A}, \; A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq A. \tag{4.5}$$

Then we have (4.4) $\Longrightarrow$ (4.5). *Conversely, if $\preceq$ is antisymmetric, then* (4.5) *implies* (4.4).

Proof. Let (4.4) be true, and suppose that (4.5) is not fulfilled. Then there is some $A \in \mathcal{A} \setminus \{\overline{A}\}$ such that $A \preceq \overline{A}$, but $\overline{A} \not\preceq A$. Because of (4.4), we obtain $\overline{A} = A$, a contradiction.

Conversely, let (4.5) be fulfilled, but suppose that (4.4) does not hold. Then there exists some $A \in \mathcal{A} \setminus \{\overline{A}\}$ with the property $A \preceq \overline{A}$. By (4.5), we get $\overline{A} \preceq A$. As $\preceq$ was assumed to be antisymmetric, this yields $\overline{A} = A$, a contradiction. $\square$

By the above results, it is possible to replace the if-condition in Algorithms 4.2 and 4.11 by $A \not\preceq A_j$ for all $A \in \mathcal{T}$ (and $A \not\preceq A_j$ for all $A \in \mathcal{U}$ in the backwards-iteration of Algorithm 4.11) under the assumption that the set order relation $\preceq$ is antisymmetric. However, among our introduced generalized set order relations, only the generalized certainly less order relation $\preceq^c_D$ is antisymmetric if D is a pointed cone (i.e., the cone D fulfills $D \cap (-D) = \{0\}$). To overcome this issue, it seems useful for the remaining set order relations to change the definition of minimal elements, given in Definition 4.1, by means of (4.4).

Notions similar to antisymmetry, that are fulfilled by $\preceq^u_D$, $\preceq^l_D$ and $\preceq^s_D$, are summarized below (see [13, Chapter 2.6.2]).

Proposition 4.18. (1) *If D is a convex cone, then $A \preceq^u_D B$ and $B \preceq^u_D A$ imply that $A - D = B - D$.*

(2) *If D is a convex cone, then $A \preceq^l_D B$ and $B \preceq^l_D A$ imply that $A + D = B + D$.*

(3) *If D is a convex cone, then $A \preceq^s_D B$ and $B \preceq^s_D A$ imply that $A - D = B - D$ and $A + D = B + D$.*

(4) *If D is a pointed cone, then the generalized certainly set order relation $\preceq^c_D$ is antisymmetric. Moreover, $A \preceq^c_D B$ and $B \preceq^c_D A$ imply that the set $A = B$ is single-valued.*

Proof. The first three assertion are obvious. Concerning the last statement, the assertions $a - b \in -D$ for all $a \in A$ and for all $b \in B$ and $a - b \in D$ for all $a \in A$ and for all $b \in B$ imply that $a = b$ all $a \in A$ and for all $b \in B$. □

Though $\preceq^u_D, \preceq^l_D, \preceq^p_D$ *and* $\preceq^s_D$ are not antisymmetric in $\mathcal{A}$, we can use Algorithm 4.11 effectively to some antisymmetric subfamily $\mathcal{A}^*$ of $\mathcal{A}$. Let $\preceq$ be one of the four relations and let $\mathcal{A}_\preceq$ be the family of all minimal elements in $\mathcal{A}$. The following algorithm creates a subfamily $\mathcal{A}^*$ of $\mathcal{A}$:

Algorithm 4.19. (Method for finding an antisymmetric subfamily $\mathcal{A}^*$ of $\mathcal{A}$)

Input: $\mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{R}^n$, set order relation $\preceq$
% initialization
$\mathcal{A}^* := \emptyset$
% iteration loop
for $i = 1 : 1 : m$ do
  if $\nexists\ A \in \{A_{i+1}, \ldots, A_m\}$ such that $A_i \preceq A$ and $A \preceq A_i$
    then $\mathcal{A}^* = \mathcal{A}^* \cup \{A_i\}$
  end if
end for
Output: $\mathcal{A}^*$

We can see that there is no pair (A_i, A_j) such that $A_i, A_j \in \mathcal{A}^*$, $i \neq j$, $A_i \preceq A_j$, and $A_j \preceq A_i$, that is, $\preceq$ is antisymmetric on $\mathcal{A}^*$. For every $A \in \mathcal{A}$, there exists $A' \in \mathcal{A}^*$ such that $A \preceq A'$ and $A' \preceq A$ by the construction of $\mathcal{A}^*$. Also, the set of minimal elements of $\mathcal{A}^*$, denoted by $\mathcal{A}^*_\preceq$, is nonempty and externally stable. From Theorem 4.12, we are able to determine $\mathcal{A}^*_\preceq$ by using Algorithm 4.11.

We also have the following result.

Theorem 4.20. *Let $\preceq$ be transitive, and*

$$\mathcal{A}_0 := \{A \in \mathcal{A} \setminus \mathcal{A}^* \mid \exists A' \in \mathcal{A}^*_\preceq : A \preceq A', A' \preceq A\}.$$

Then we have the following property:

$$\mathcal{A}^*_{\preceq} \cup \mathcal{A}_0 = \mathcal{A}_{\preceq}.$$

Proof. If $\bar{A} \in \mathcal{A}_{\preceq}$, there exists $A' \in \mathcal{A}^*$ such that $\bar{A} \preceq A'$ and $A' \preceq \bar{A}$. We obtain $A' \in \mathcal{A}_{\preceq}$. Since $A' \in \mathcal{A}^*$ and $\mathcal{A}^* \subset \mathcal{A}$, $A' \in \mathcal{A}^*_{\preceq}$. Hence, if $\bar{A} \in \mathcal{A} \setminus \mathcal{A}^*$, then $\bar{A} \in \mathcal{A}_0$. If $\bar{A} \in \mathcal{A}^*$, from the antisymmetry on $\mathcal{A}^*$, $\bar{A} = A' \in \mathcal{A}^*_{\preceq}$. Conversely, assume that $\bar{A} \in \mathcal{A}^*_{\preceq} \cup \mathcal{A}_0$. If $\bar{A} \notin \mathcal{A}_{\preceq}$, there exists $A \in \mathcal{A}$ such that $A \preceq \bar{A}$ and $\bar{A} \not\preceq A$. Also there exists $A^* \in \mathcal{A}^*$ such that $A \preceq A^*$ and $A^* \preceq A$. From the transitivity, we have $A^* \preceq \bar{A}$ and $\bar{A} \not\preceq A^*$. This shows that $\bar{A} \notin \mathcal{A}^*_{\preceq}$. Then $\bar{A} \in \mathcal{A}_0$, and there exists $A' \in \mathcal{A}^*_{\preceq}$ such that $A' \preceq \bar{A}$ and $\bar{A} \preceq A'$. From the transitivity, we have $A^* \preceq A'$ and $A' \not\preceq A^*$. This contradicts with $A^* \in \mathcal{A}^*$ and $A' \in \mathcal{A}^*_{\preceq}$. □

Finally, we propose the following algorithm that does not rely on antisymmetry or external stability of the set order relation $\preceq$. The idea stems from Eichfelder [3, Algorithm 1], who gave a similar numerical procedure for finding minimal elements in vector optimization with a variable ordering structure. In the following algorithm, a third for-loop is added which compares the elements that were obtained in the set $\mathcal{U}$ by Algorithm 4.11 with all remaining elements in $\mathcal{A} \setminus \mathcal{U}$.

Algorithm 4.21. (Jahn-Graef-Younes method with backward iteration for finding minimal elements of a family of finitely many sets)

Input: $\mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{R}^n$, set order relation $\preceq$
% initialization
$\mathcal{T} := \{A_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ **do**
  if $(A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
  end if
end for
$\{A_1, \ldots, A_p\} := \mathcal{T}$
$\mathcal{U} := \{A_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ **do**
  if $(A \preceq A_j,\ A \in \mathcal{U} \implies A_j \preceq A)$ **then**
    $\mathcal{U} := \mathcal{U} \cup \{A_j\}$
  end if
end for
$\{A_1, \ldots, A_q\} := \mathcal{U}$
$\mathcal{V} := \emptyset$
% final comparison
for $j = 1 : 1 : q$ **do**
  if $(A \preceq A_j,\ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq A)$ **then**
    $\mathcal{V} := \mathcal{V} \cup \{A_j\}$
  end if

end for
Output: $\mathcal{V}$

Theorem 4.22. *Algorithm 4.21 consists of exactly all minimal elements of the family of sets $\mathcal{A}$.*

Proof. Let A_j be an arbitrary element in $\mathcal{V}$. Then $A_j \in \mathcal{U}$, as $\mathcal{V} \subseteq \mathcal{U}$, and

$$A \preceq A_j, \ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq A.$$

Suppose that A_j is not minimal in $\mathcal{A}$. Then, there exists some $A \in \mathcal{A}$ such that $A \preceq A_j$ and $A_j \not\preceq A$. If $A \notin \mathcal{U}$, then this is a contradiction. If $A \in \mathcal{U}$, then A is also minimal in $\mathcal{U}$ (compare the proof of Theorem 4.12). Since $A_j \in \mathcal{U}$, and A_j is also minimal in $\mathcal{U}$, we obtain from $A \preceq A_j$ that $A_j \preceq A$, a contradiction.

Conversely, let A_j be minimal in $\mathcal{A}$. Then we get

$$A \preceq A_j, \ A \in \mathcal{A} \implies A_j \preceq A.$$

Now, suppose that $A_j \notin \mathcal{V}$. Then there exists some $A \in \mathcal{A} \setminus \mathcal{U}$ with $A \preceq A_j$ and $A_j \not\preceq A$. As A_j is minimal in $\mathcal{A}$, we get $A_j \preceq A$, a contradiction. □

Remark 4.23. Note that it is again possible to evaluate the implication

$$A \preceq A_j, \ A \in \mathcal{T} \ (\mathcal{U}, \ \mathcal{A} \setminus \mathcal{U}, \text{ resp.}) \implies A_j \preceq A$$

in Algorithm 4.21 by means of the nonlinear scalarizing functional $z^{D,k}$. This can be done analogously to the proposed process on page 55, but we refrain from repeating it here due to its similarities.

Example 4.24. Let $D := \mathbb{R}^2_+$ and $\preceq := \preceq^u_D$. We use the same family of randomly computed sets from Example 4.7. Out of the considered 1.000 sets, a total number of 5 are minimal w.r.t. to $\preceq$. Algorithm 4.21 first generates 18 sets in $\mathcal{T}$, which is already a huge reduction, and finally collects all minimal elements within the set $\mathcal{U}$, which coincides with $\mathcal{V}$. In Figure 3 the minimal elements are darkly filled, while the lighter sets are those elements that are not minimal, but belong to the set $\mathcal{T}$. Of course, in our case the set of minimal elements is externally stable because of the unified structure of the sets.

Example 4.25. Let $D := \mathbb{R}^2_+$, $\preceq := \preceq^p_D$, $A_1 := \{(0,0)\}$, $A_2 := \{(1,1),(2,-1)\}$, $A_3 := \{(3,-0.5)\}$. The family of sets is given as $\mathcal{A} := \{A_1, A_2, A_3\}$. The only minimal element of $\mathcal{A}$ w.r.t. $\preceq$ is $A_1 = \{0,0\}$. Algorithm 4.2 generates the sets $\mathcal{T} := \{A_1, A_3\}$ and $\mathcal{U} = \{A_3, A_1\}$. A final comparison then yields $\mathcal{V} = \{A_1\}$.

Remark 4.26. A finite family of sets $\mathcal{A}$ can also be computed by an appropriate discretization of the outcome sets of the considered (continuous) set optimization problem.

5. Conclusions

In this paper, we introduced very general set order relations and characterized them based on a prominent scalarizing functional from vector optimization. We moreover proposed a numerical algorithm that reduces the numerical effort while sorting out non-minimal elements of a family of sets and extended this method to

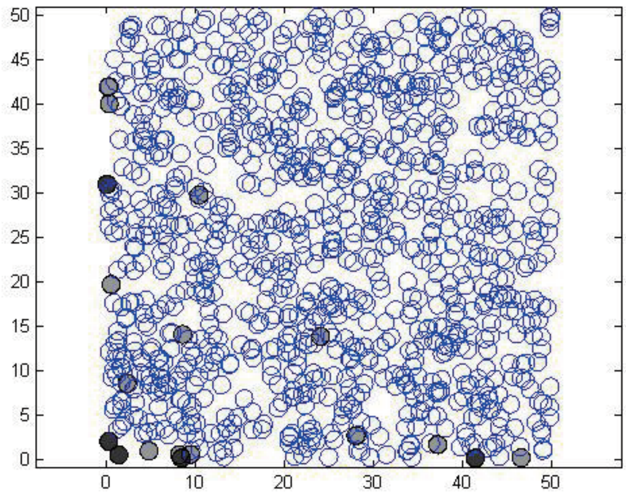

FIGURE 3. The lightly filled circles belong to the set $\mathcal{T}$ generated by Algorithm 4.21 and the darkly filled circles are the elements which are minimal w.r.t. $\preceq_D^u$ (see Example 4.24).

select the sets which are minimal. Our approach can be regarded as an extension of the well-known Jahn-Graef-Younes method. More research shall be done on the implementations of Algorithms 4.2, 4.11, 4.19 and 4.21 to specific applications of set optimization problems.

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