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A NEW PARAMETRIC KERNEL FUNCTION WITH TRIGONOMETRIC BARRIER TERM FOR CONVEX QUADRATIC SYMMETRIC CONE OPTIMIZATION

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Abstract. In this paper, a new parametric kernel function with trigonometric barrier term for convex quadratic optimization over symmetric cone is considered. By employing Euclidean Jordan algebras, we obtain the iteration bound for primal-dual large-update interior-point methods, namely, $O\left(\sqrt{r}\log r \log \frac{r}{\varepsilon}\right)$, where r is the rank of Euclidean Jordan algebra and ε is the desired accuracy in terms of the objective value. The obtained iteration bound is as good as the currently best known iteration bounds for these type methods.

1. Introduction

Recall that a Euclidean Jordan algebra (EJA) is said to be simple if it cannot be represented as the orthogonal direct sum of two EJAs. Let $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ ($\mathcal{V}$, in short) be a EJA with rank r and $\mathcal{K}$ be the corresponding symmetric cone, $\langle\cdot,\cdot\rangle$ denotes the inner product. More definitions and properties will be presented explicitly in Section 2. For a comprehensive treatment of EJAs, see also [13].

In this paper, we focus on the primal problem of convex quadratic symmetric cone optimization (CQSCO) [21] in the form

$$\begin{aligned} (P) \quad \min \quad & f(x) = \frac{1}{2}\langle x, Q(x)\rangle + \langle c, x\rangle \\ \text{s.t.} \quad & \mathcal{A}(x) = b,\ x \in \mathcal{K}, \end{aligned}$$

and its dual problem

$$\begin{aligned} (D) \quad \max \quad & -\frac{1}{2}\langle x, Q(x)\rangle + b^T y \\ \text{s.t.} \quad & \mathcal{A}^T y + s = \nabla f(x) = Q(x) + c,\ s \in \mathcal{K},\ y \in \mathbf{R}^m, \end{aligned}$$

Here, $c \in \mathcal{V}$ and $b \in \mathbf{R}^m$ are given data, $\mathcal{A} : \mathcal{V} \to \mathbf{R}^m$ is a given linear map, $\mathcal{A}^T$ is the adjoint of $\mathcal{A}$, and Q is a given self-adjoint positive semidefinite (with respect to $\langle\cdot,\cdot\rangle$) linear operator on $\mathcal{V}$, i.e., for any $x, s \in \mathcal{V}$, then $\langle Q(x), s\rangle = \langle x, Q(s)\rangle$ and $\langle Q(x), x\rangle \geq 0$. Throughout the paper, we assume that the linear map $\mathcal{A}$ is surjective, which implies that $\mathcal{A}\mathcal{A}^T$ is nonsingular.

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Since the groundbreaking paper of Karmarkar, many researchers have proposed and analyzed various interior-point methods (IPMs) for linear optimization (LO) and a large amount of results have been reported. In the literature, the Newton direction is used as the search direction in most primal-dual IPMs based on the logarithmic barrier function. However, there is still a gap between the practical behavior of these algorithms and these theoretical performance results [25]. For a survey, we refer to the monograph [26, 28] on the subject and the references therein.

Peng et al. [25] considered the self-regular barrier function, which is fairly general and includes the logarithmic barrier function as a special case. The iteration bounds of large- and small-update methods for LO based on the self-regular barriers are obtained, namely, $O\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$ and $O\left(\sqrt{n}\log \frac{n}{\varepsilon}\right)$, respectively, where n denotes the number of inequalities in the problem, and ϵ denotes the desired accuracy in terms of the objective value. Bai et al. [4] introduced a variety of non-self-regular kernel functions, i.e., the so-called eligible kernel functions, which defined by some simple conditions on the kernel functions and their derivatives. They provided a simple and unified computational scheme for the complexity analysis of primal-dual kernel function based IPMs for LO. Subsequently, a series of eligible kernel functions are considered for various optimization problems and complementarity problems, see, e.g., [2, 5, 6, 8, 9, 14, 15, 18, 19, 22–24, 29, 30, 32, 33].

Jordan algebras were created to illuminate a particular aspect of physics: the quantum mechanical observably. However, Jordan algebras illuminated connections with many other areas of mathematics. Specially their relation to symmetric cones. In fact any symmetric cone, can be realized as a cone of squares of some EJA. It turns out that EJAs provided the tools to treat optimization problems involving symmetric cones: with a simple structure to analyze, at once, all symmetric optimization problems [16, 29, 31].

There is an extensive literature on the analysis of the optimization problems over symmetric cones due to EJA tool. Faybusovich [11] made the first attempt to extend IPMs from semidefinite optimization (SDO) to symmetric cone optimization (SCO) by using EJAs. Schmieta and Alizadeh [27] provided a unified method of the analysis for many IPMs in symmetric cones extensively under the framework of EJAs. Since then, several IPMs designed for LO, second-order cone optimization (SOCO) and SDO have been successfully extended to SCO and CQSCO. For a survey, we refer to the monograph [1] on the subject and the references therein.

Recently, El Ghami et al. [15] first introduced a trigonometric kernel function as follows

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{6}{\pi}\tan\left(\frac{\pi(1-t)}{2+4t}\right).$$

They established the worst case iteration bounds of large- and small-update methods for LO, namely, $O(n^{\frac{3}{4}}\log \frac{n}{\varepsilon})$ and $O(\sqrt{n}\log \frac{n}{\varepsilon})$, respectively. Consequently, a class of kernel functions with trigonometric barrier terms are discovered [8, 9, 18, 19, 22–24]. The corresponding iteration bounds for large-update methods are collected in Table 1. In most cases, the complexity results for small-update IPMs are essentially the same small-update methods based on the classic logarithmic barrier function, which is $O(\sqrt{n}\log \frac{n}{\varepsilon})$.

i	The kernel functions $\psi_i(t)$	Large-update methods	Ref.
1	$\frac{t^2-1}{2} + \frac{6}{\pi}\tan\left(\frac{\pi(1-t)}{2+4t}\right)$	$O\left(n^{\frac{3}{4}} \log\frac{n}{\varepsilon}\right)$	[15]
2	$\frac{t^2-1}{2} - \log t + \frac{1}{8}\tan^2\left(\frac{\pi(1-t)}{2+4t}\right)$	$O\left(n^{\frac{2}{3}} \log\frac{n}{\varepsilon}\right)$	[23]
3	$\frac{t^2-1}{2} - \int_1^t e^{3(\tan(\frac{\pi}{2+2\xi})-1)}d\xi$	$O\left(\sqrt{n}(\log n)^2 \log\frac{n}{\varepsilon}\right)$	[24]
4	$\frac{t^2-1}{2} - \log t + \lambda\tan^2\left(\frac{\pi(1-t)}{2+3t}\right), 0 < \lambda \leq \frac{8}{25\pi}$	$O\left(n^{\frac{3}{4}} \log\frac{n}{\varepsilon}\right)$	[9]
5	$t^2 - 2t + \frac{1}{\sin\left(\frac{\pi t}{1+t}\right)}$	$O\left(n^{\frac{3}{4}} \log\frac{n}{\varepsilon}\right)$	[18]
6	$\frac{t^2-1}{2} + \frac{4}{\pi}\cot\left(\frac{\pi t}{1+t}\right)$	$O\left(n^{\frac{3}{4}} \log\frac{n}{\varepsilon}\right)$	[19]
7	$\frac{(t-1)^2}{2} + \frac{(t-1)^2}{2t} + \frac{1}{8}\left(\tan^2\left(\frac{\pi(1-t)}{2+4t}\right)\right)$	$O\left(n^{\frac{2}{3}} \log\frac{n}{\varepsilon}\right)$	[22]
8	$\frac{t^2-1}{2} - \frac{4}{\pi p}\left(\tan^p\left(\frac{\pi}{2+2t}\right) - 1\right), p \geq 2$	$O\left(pn^{\frac{p+2}{2(p+1)}} \log\frac{n}{\varepsilon}\right)$	[8]

Table 1. Complexity results for kernel functions with trigonometric barrier terms

The purpose of the paper is to present a class of primal-dual large-update IPMs for CQSCO based on a new kind of parametric kernel function with trigonometric barrier term. By employing EJAs, we derive the currently best result of iteration bounds for these type methods.

The outline of the paper is as follows. In Section 2, we provide the basic analysis on symmetric cone and some properties of the parametric kernel (and barrier) function. In Section 3, primal-dual IPMs for CQSCO based on the parametric kernel function are presented. In Section 4, we give some conclusions.

2. Preliminaries

In this section, we present some basic results on EJAs, and properties of the new parametric barrier function that are needed in the analysis of our algorithm.

2.1. **Analysis on symmetric cone.** We give some basic analysis on symmetric cone by using EJAs. A comprehensive treatment of EJAs can be found in the monograph [13] and the references [11, 16, 27, 29].

Let $x, y \in \mathcal{V}$. Then the Lyapunov transformation $L(x)$ is defined by

$$L(x)y := x \circ y. \tag{2.1}$$

Furthermore, the quadratic representation $P(x)$ is given by

$$P(x) := 2L(x)^2 - L(x^2), \tag{2.2}$$

where $L(x)^2 = L(x)L(x)$.

For any EJA $\mathcal{V}$, the corresponding cone of squares

$$\mathcal{K}(\mathcal{V}) := \{x^2 : x \in \mathcal{V}\} \tag{2.3}$$

is indeed a symmetric cone (see, Theorem III.2.1 in [13]). In the sequel, $\mathcal{K}$ will always denote a symmetric cone, and $\mathcal{V}$ an EJA with $\operatorname{rank}(\mathcal{V}) = r$ for which $\mathcal{K}$ is its cone of squares.

The following theorem gives an important decomposition, the spectral decomposition, on the space $\mathcal{V}$.

Theorem 2.1 (Theorem III.1.2 in [13])**.** *Let $x \in \mathcal{V}$. Then there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$ such that*

$$x = \sum_{i=1}^{r} \lambda_i(x) c_i. \tag{2.4}$$

The numbers $\lambda_i(x)$ (with their multiplicities) are called the eigenvalues of x. Furthermore, the trace and the determinant of x are given by

$$\operatorname{tr}(x) = \sum_{i=1}^{r} \lambda_i(x), \text{ and } \det(x) = \prod_{i=1}^{r} \lambda_i(x),$$

respectively.

Let $x \in \mathcal{V}$ with the spectral decomposition given by (2.4), the vector-valued function $\psi(x)$ is defined by

$$\psi(x) := \psi(\lambda_1(x))\, c_1 + \cdots + \psi(\lambda_r(x))\, c_r. \tag{2.5}$$

Furthermore, if $\psi(t)$ is differentiable, the derivative $\psi'(t)$ exist, and we also have the vector-valued function $\psi'(x)$, namely

$$\psi'(x) = \psi'(\lambda_1(x))\, c_1 + \cdots + \psi'(\lambda_r(x))\, c_r. \tag{2.6}$$

It should be noted that $\psi'(x)$ is just a vector valued function induced by the derivative $\psi'(t)$ of the function $\psi(t)$ rather than the derivative of the vector valued function $\psi(x)$ defined by (2.5).

The following theorem provides another important decomposition, the Peirce decomposition, on the space $\mathcal{V}$.

Theorem 2.2 (Theorem IV.2.1 in [13])**.** *Let $x \in \mathcal{V}$ with the spectral decomposition given by (2.4). Then we have*

$$\mathcal{V} = \oplus_{i \le j} \mathcal{V}_{ij},$$

where

$$\mathcal{V}_{ii} := \{x | x \circ c_i = x\}, \text{ and } \mathcal{V}_{ij} := \left\{ x | x \circ c_i = \frac{1}{2} x = x \circ c_j \right\}, \; 1 \le i < j \le r,$$

are Pierce spaces of $\mathcal{V}$. Then, for any $x \in \mathcal{V}$, there exists $x_i \in \mathbf{R}$, $c_i \in \mathcal{V}_{ii}$ and $x_{ij} \in \mathcal{V}_{ij}$ $(i < j)$ such that

$$x = \sum_{i=1}^{r} x_i c_i + \sum_{i<j} x_{ij}.$$

Let $x, s \in \mathcal{V}$. The trace inner product is defined by

$$\langle x, s\rangle := \mathrm{tr}(x \circ s). \tag{2.7}$$

Then the Frobenius norm, namely $\|\cdot\|_F$, is given by

$$\|x\|_F := \sqrt{\langle x, x\rangle} = \sqrt{\mathrm{tr}(x^2)} = \sqrt{\sum_{i=1}^{r} \lambda_i^2(x)}. \tag{2.8}$$

It follows that

$$|\lambda_{min}(x)| \leq \|x\|_F, \text{ and } |\lambda_{max}(x)| \leq \|x\|_F. \tag{2.9}$$

The following lemma gives the so-called NT-scaling of $\mathcal{V}$, which plays an important role in the analysis of the algorithm presented in Fig. 1.

Lemma 2.3 (Lemma 3.2 in [12])**.** *Let $x, s \in \mathrm{int}\mathcal{K}$. Then there exists a unique $w \in \mathrm{int}\mathcal{K}$ such that*

$$x = P(w)s.$$

Moreover,

$$w = P(x)^{\frac{1}{2}} \left(P(x^{\frac{1}{2}})s\right)^{-\frac{1}{2}} \left[= P(s^{-\frac{1}{2}}) \left(P(s^{\frac{1}{2}})x\right)^{\frac{1}{2}}\right].$$

The point w is called the scaling point of x and s (in this order).

Recall that two elements x and s in $\mathcal{V}$ are similar, and briefly denoted as $x \sim s$, if x and s share the same eigenvalues, including their multiplicities (see, e.g., [27,29]).

Lemma 2.4 (Proposition 21 in [27] and Proposition 3.3 in [29])**.** *Let $x, s, z \in \mathrm{int}\mathcal{K}$ and w be the scaling point of x and s. Then*
(i) $(P(x^{1/2})s)^{1/2} \sim P(w^{1/2})s$;
(ii) $P(x^{1/2})s \sim P(s^{1/2})x$;
(iii) $P(x^{1/2})s \sim P((P(z)x)^{1/2})P(z^{-1})s$.

Let $x = \sum_{i=1}^{r} \lambda_i(x)c_i$ be the spectral decomposition of $x \in \mathcal{V}$ with respect to the Jordan frame $\{c_1, \ldots, c_r\}$. The following two theorems give explicitly the first derivatives of the real valued separable spectral function $F : \mathcal{V} \to \mathbf{R}$ and the vector valued separable spectral function $G : \mathcal{V} \to \mathcal{V}$, respectively.

Theorem 2.5 (Theorem 38 in [7])**.** *If f is continuously differentiable function in a suitable domain that contains all the eigenvalues of x, then the real valued separable spectral function*

$$F(x) := \sum_{i=1}^{r} f(\lambda_i(x)), \tag{2.10}$$

is continuously differentiable at x and

$$D_x F(x) = \sum_{i=1}^{r} f'(\lambda_i(x))c_i.$$

Theorem 2.6 (Lemma 1 in [20]). *If g is continuously differentiable function in a suitable domain that contains all the eigenvalues of x, then the vector valued separable spectral function*

$$G(x) := \sum_{i=1}^{r} g(\lambda_i(x))c_i, \tag{2.11}$$

is continuously differentiable at x and

$$D_x G(x) = \sum_{i=1}^{r} g'(\lambda_i(x))P_{ii} + \sum_{j<k} \frac{g(\lambda_j(x)) - g(\lambda_k(x))}{\lambda_j(x) - \lambda_k(x)} P_{jk},$$

where $P_{ii} = P(c_i)$ and $P_{jk} = 4L(c_j)L(c_k)$. When $\lambda_j(x) = \lambda_k(x)$, the quotient is understood as the second derivative of $g(\lambda_j(x))$, i.e., $g''(\lambda_j(x))$.

2.2. **Properties of the parametric kernel (barrier) function.** In this paper, we consider a kind of parametric kernel function with trigonometric barrier term as follows

$$\psi(t) := \frac{t^2-1}{2} - \log t - \int_1^t \frac{u^2}{2p(x+2u)^2} \tan^{2p}(h(x))\mathrm{d}x, \ t > 0, \ p \in \mathbf{N}, \ p > 1, \tag{2.12}$$

where

$$h(x) = \frac{\pi u(1-x)}{x+2u} \tag{2.13}$$

and $0 < u \leq u^*$, ($u^* \approx 0.4275$). Also, u^* is the unique solution of the equation

$$g(u) := \tan\left(\frac{(1-2u)\pi}{4}\right) - \frac{2}{3\pi(1+2u)} = 0. \tag{2.14}$$

This parametric kernel function is fairly general and includes the classic barrier function as a special case when $\mu = 0$.

The first three derivatives of $\psi(t)$ with respect to t are given by

$$\psi'(t) = t - \frac{1}{t} - \frac{u^2}{2p(t+2u)^2} \tan^{2p}(h(t)), \tag{2.15}$$

$$\begin{aligned}\psi''(t) = {} & 1 + \frac{1}{t^2} + \frac{u^2}{p(t+2u)^3} \tan^{2p}(h(t)) \\ & + \frac{\pi u^3(1+2u)}{(t+2u)^4} \tan^{2p-1}(h(t)) \sec^2(h(t)),\end{aligned} \tag{2.16}$$

$$\begin{aligned}\psi'''(t) = {} & -\frac{2}{t^3} - \frac{3u^2}{p(t+2u)^4} \tan^{2p}(h(t)) \\ & - \frac{6\pi u^3(1+2u)}{(t+2u)^5} \tan^{2p-1}(h(t)) \sec^2(h(t)) \\ & - \frac{\pi^2 u^4(1+2u)^2(2p-1)}{(t+2u)^6} \tan^{2p-2}(h(t)) \sec^4(h(t))\end{aligned} \tag{2.17}$$

$$-\frac{2\pi u^4(1+2u)^2}{(t+2u)^6}\tan^{2p}(h(t))\sec^2(h(t)).$$

One can easily verify that

$$\psi(1)=\psi'(1)=0,\ \text{and}\ \lim_{t\to 0^+}\psi(t)=\lim_{t\to+\infty}\psi(t)=+\infty.$$

Moreover, the proposed kernel function $\psi(t)$ is completely defined by its second derivative, i.e.,

$$\psi(t)=\int_1^t\int_1^\xi \psi''(\zeta)d\zeta d\xi.$$

The following lemma provides some properties of the parametric kernel function $\psi(t)$ defined by (2.12).

Lemma 2.7. *One has*

$$\psi''(t)>1,\ \forall t>0; \tag{18-a}$$

$$t\psi''(t)+\psi'(t)>0,\ \forall t>0; \tag{18-b}$$

$$t\psi''(t)-\psi'(t)>0,\ \forall t>1; \tag{18-c}$$

$$\psi'''(t)<0,\ \forall t>0. \tag{18-d}$$

Proof. See Appendix A.

To analyze the algorithms, we define the barrier function $\Psi(v):\text{int}\mathcal{K}\to\mathbf{R}_+$ based on the proposed parametric kernel function as follows

$$\Psi(v):=\sum_{i=1}^n\psi(\lambda_i(v)). \tag{2.19}$$

One can conclude that $\Psi(v)$ is nonnegative, strict convex and vanishes if and only if $v=e$ (see, e.g., [4, 25]). Furthermore, we define the norm-based proximity $\delta(v)$: $\text{int}\mathcal{K}\to\mathbf{R}_+$ as follows

$$\delta(v):=\frac{1}{2}\|\nabla\Psi(v)\|_F=\frac{1}{2}\sqrt{\sum_{i=1}^r\psi'(\lambda_i(v))^2}. \tag{2.20}$$

One can easily verify that $\delta(v)\geq 0$, and $\delta(v)=0$ if and only if $\Psi(v)=0$.

The proposed parametric kernel function $\psi(t)$ is strongly convex due to the fact that (18-a) of Lemma 2.7, i.e., $\psi''(t)>1$. As a result, we have the following lemma (see, e.g., Lemma 2.1 in [3]).

Lemma 2.8. *If* $t>0$*, then*

$$\frac{1}{2}(t-1)^2\leq\psi(t)\leq\frac{1}{2}\psi'(t)^2.$$

The following lemma provides an upper bound on $\|v\|$ in terms of $\Psi(v)$ and r.

Lemma 2.9. *If* $\Psi(v)\geq 1$*, then*

$$\|v\|_F\leq\sqrt{r}+\sqrt{2\Psi(v)}.$$

Proof. From Lemma 2.8, we have

$$t \leq 1 + \sqrt{2\psi(t)}.$$

This implies that

$$\lambda_i(v) \leq 1 + \sqrt{2\psi(\lambda_i(v))},\ i = 1, \cdots, r.$$

Then

$$\|v\|_F^2 = \sum_{i=1}^r \lambda_i(v)^2 \leq \sum_{i=1}^r (1 + \sqrt{2\psi(\lambda_i(v))})^2 \leq \sum_{i=1}^r (1 + 4\psi(\lambda_i(v))) \leq r + 4\Psi(v).$$

Hence, we have

$$\|v\|_F \leq \sqrt{r} + \sqrt{2\Psi(v)}.$$

This completes the proof. □

The following lemma shows that the proposed parametric kernel function $\psi(t)$ is exponential convex.

Lemma 2.10. (Lemma 2.1 in [4]) *Let $\psi(t)$ be a twice differentiable function for $t > 0$. Then the following three properties are equivalent:*
(i) $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$ for $t_1, t_2 \geq 0$;
(ii) $t\psi''(t) + \psi'(t) > 0$ for $t > 0$;
(iii) $\psi(e^{\xi})$ is convex.

As a consequence of Lemma 2.10, we have the following important result.

Theorem 2.11 (Theorem 4.3.2 in [29])**.** *If $x, s \in \mathrm{int}\mathcal{K}$, then*

$$\Psi\left((P(x)^{1/2}s)^{1/2}\right) \leq \frac{1}{2}(\Psi(x) + \Psi(s)).$$

In what follows, we want to find an upper bound for $\Psi(\beta v)$ with $\beta = \frac{1}{\sqrt{1-\theta}}$ in terms of $\Psi(v)$. We start with the following lemma.

Lemma 2.12. *If $\beta \geq 1$, then*

$$\psi(\beta t) \leq \psi(t) + \frac{1}{2}(\beta^2 - 1)t^2.$$

Proof. Let

$$\psi_b(t) := -\log t - \int_1^t \frac{u^2}{2p(x+2u)^2}\tan^{2p}(h(x))\mathrm{d}x,\ 0 < u \leq u^*.$$

Then

$$\psi(t) = \frac{t^2 - 1}{2} + \psi_b(t). \tag{2.21}$$

Furthermore, we have

$$\psi(\beta t) - \psi(t) = \frac{1}{2}(\beta^2 - 1)t^2 + \psi_b(\beta t) - \psi_b(t).$$

Note that $\beta \geq 1$, to prove the lemma, it is sufficient to show that the function $\psi_b(t)$ is a decreasing function. This due to the following fact that

$$\psi_b'(t) = -\frac{1}{t} - \frac{u^2}{2p(t+2u)^2}\tan^{2p}(h(t)) < 0.$$

This completes the proof. □

Theorem 2.13. *If* $0 < \theta < 1$ *and* $v_+ = \frac{v}{\sqrt{1-\theta}}$, *then*

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)} \left(2\Psi(v) + 2\sqrt{2r\Psi(v)} + r\right).$$

Proof. From Lemma 2.12 with $\beta = \frac{1}{\sqrt{1-\theta}}$, we have

$$\Psi(\beta v) \leq \Psi(v) + \frac{1}{2}\sum_{i=1}^{n}(\beta^2 - 1)v_i^2 = \Psi(v) + \frac{\theta\|v\|_F^2}{2(1-\theta)}.$$

Lemma 2.9 implies that

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)} \left(2\Psi(v) + 2\sqrt{2r\Psi(v)} + r\right).$$

This complete the proof. □

Let $x(t) = x_0 + tu \in \mathrm{int}\mathcal{K}$ with $t \in \mathbf{R}$ and $u \in \mathcal{V}$. From Theorem 2.1, the spectral decomposition of $x(t)$ with respect to the Jordan frame $\{c_1, \ldots, c_r\}$ can be defined by

$$x(t) = \sum_{i=1}^{r} \lambda_i(x(t))c_i. \tag{2.22}$$

Similarly, we have the Pierce decomposition of u, by Theorem 2.2,

$$u = \sum_{i=1}^{r} u_i c_i + \sum_{i<j} u_{ij}. \tag{2.23}$$

It follows from Theorem 2.5 and Theorem 2.6 that the first two derivatives of the general function $\Psi(x(t))$ with respect to t are given by (see, e.g., [29])

$$D_t\Psi(x(t)) = \mathrm{tr}(D_x\Psi(x(t)) \circ x'(t)) = \mathrm{tr}\left(\sum_{i=1}^{r} \psi'(\lambda_i(x(t)))c_i \circ u\right), \tag{2.24}$$

and

(2.25)

$$D_t^2\Psi(x(t)) = \sum_{i=1}^{r} \psi''(\lambda_i(x(t)))(u_i)^2 + \sum_{j<k} \frac{\psi'(\lambda_j(x(t))) - \psi'(\lambda_k(x(t)))}{\lambda_j(x(t)) - \lambda_k(x(t))}\mathrm{tr}((u_{jk})^2).$$

When $\lambda_j(x(t)) = \lambda_k(x(t))$, the quotient is understood as the second derivative of $\psi(\lambda_j(x(t)))$, i.e., $\psi''(\lambda_j(x(t)))$.

Recall that $\psi''(t)$ is monotonically decreasing in $t \in (0, +\infty)$ according to (18-d). Under the assumption that $j < k$ implies $\lambda_j(x(t)) \geq \lambda_k(x(t))$, we can conclude that

$$D_t^2\Psi(x(t)) \leq \sum_{i=1}^{r} \psi''(\lambda_i(x(t)))(u_i)^2 + \sum_{j<k} \psi''(\lambda_k(x(t)))\mathrm{tr}((u_{jk})^2), \tag{2.26}$$

which bounds the second-order derivative of $\Psi(x(t))$ with respect to t.

3. Kernel function-based IPMs for CQSCO

3.1. **Framework of the kernel function-based IPMs.** Without loss of generality, we assume that both (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (x^0, y^0, s^0) such that

$$\mathcal{A}(x^0) = b,\ x^0 \succ_{\mathcal{K}} 0,\ \mathcal{A}^T y^0 + s^0 - Q(x^0) = c,\ s^0 \succ_{\mathcal{K}} 0.$$

Then the Karush-Kuhn-Tucker optimality condition for CQSCO under the IPC is equivalent to solve the following nonlinear system

$$\begin{aligned} \mathcal{A}(x) &= b,\ x \succeq_{\mathcal{K}} 0, \\ \mathcal{A}^T y + s - Q(x) &= c,\ s \succeq_{\mathcal{K}} 0, \\ x \circ s &= 0. \end{aligned} \tag{3.1}$$

Replacing the third equation of the system (3.1) by the nonlinear parameterized equation $x \circ s = \mu e$ with $\mu > 0$, we have

$$\begin{aligned} \mathcal{A}(x) &= b,\ x \succeq_{\mathcal{K}} 0, \\ \mathcal{A}^T y + s - Q(x) &= c,\ s \succeq_{\mathcal{K}} 0, \\ x \circ s &= \mu e. \end{aligned} \tag{3.2}$$

Since the IPC holds and $\mathcal{A}$ is surjective, the nonlinear parameterized system (3.2) has a unique solution $(x(\mu), y(\mu), s(\mu))$ for each $\mu > 0$, and we call $x(\mu)$ the μ-center of (P) and $(y(\mu), s(\mu))$ the μ-center of (D) [11,21]. The set of μ-centers gives a homotopy path (with μ running through all the positive real numbers), which is called the central path. If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition $x \circ s = 0$, it naturally yields an optimal solution for (P) and (D) (see, e.g., [11,21,27]).

Applying Newton's method to the nonlinear parametric system (3.2), we have

$$\begin{aligned} \mathcal{A}(\Delta x) &= 0, \\ \mathcal{A}^T \Delta y + \Delta s - Q(\Delta x) &= 0, \\ x \circ \Delta s + s \circ \Delta x &= \mu e - x \circ s. \end{aligned} \tag{3.3}$$

Due to the fact that x and s do not operator commute in general, i.e., $L(x)L(s) \neq L(s)L(x)$, this system not always have a unique solution. It is well known that this difficulty can be solved by applying the following scaling scheme (see, Lemma 28 in [27])

$$x \circ s = \mu e \ \Leftrightarrow\ P(u)x \circ P(u)^{-1}s = \mu e,\ u \in \mathrm{int}\mathcal{K}.$$

Then we can consider the following nonlinear parametric system

$$\begin{aligned} \mathcal{A}(x) &= b,\ x \succeq_{\mathcal{K}} 0, \\ \mathcal{A}^T y + s - Q(x) &= c,\ s \succeq_{\mathcal{K}} 0, \\ P(u)x \circ P(u^{-1})s &= \mu e. \end{aligned} \tag{3.4}$$

Applying Newton's method to the nonlinear parametric system (3.4), we have

$$\begin{aligned} \mathcal{A}(\Delta x) &= 0, \\ \mathcal{A}^T \Delta y + \Delta s - Q(\Delta x) &= 0, \end{aligned} \tag{3.5}$$

$$P(u)x \circ P(u)^{-1}\Delta s + P(u)^{-1}s \circ P(u)\Delta x = \mu e - P(u)x \circ P(u^{-1})s.$$

Let $u = w^{-\frac{1}{2}}$, where w is the NT-scaling point of x and s. We define

$$v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \left[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}} \right], \tag{3.6}$$

and

$$\overline{\mathcal{A}} := \frac{\mathcal{A}P(w)^{\frac{1}{2}}}{\sqrt{\mu}},\ d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}},\ d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \tag{3.7}$$

From (3.6) and (3.7), after some elementary reductions, we have

$$\begin{aligned} \overline{\mathcal{A}}(d_x) &= 0, \\ \overline{\mathcal{A}}^T \Delta y + d_s - \overline{Q}(d_x) &= 0, \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{3.8}$$

where $\overline{Q} = P(w)^{\frac{1}{2}}QP(w)^{\frac{1}{2}}$. The system has a unique solution (see, e.g., [11, 21]).

Considering the classical logarithmic function as follows

$$\psi_c(t) = \frac{t^2 - 1}{2} - \log t, \tag{3.9}$$

we have

$$v^{-1} - v = -\nabla\Psi_c(v),$$

where $\nabla\Psi_c(v)$ is the gradient of the classical logarithmic barrier function

$$\Psi_c(v) := \operatorname{tr}(\psi_c(v)) = \sum_{i=1}^{r} (\psi_c(\lambda_i(v)). \tag{3.10}$$

Then the system (3.8) is equivalent to the following system

$$\begin{aligned} \overline{\mathcal{A}}(d_x) &= 0, \\ \overline{\mathcal{A}}^T \Delta y + d_s - \overline{Q}(d_x) &= 0, \\ d_x + d_s &= -\nabla\Psi_c(v). \end{aligned} \tag{3.11}$$

Follows the strategy considered in [4], we rewrite the barrier function $\Psi(v)$ defined by (2.19) as follows.

$$\Psi(x, s; \mu) := \Psi(v) := \sum_{i=1}^{n} \psi(\lambda_i(v)). \tag{3.12}$$

Replacing the right-hand side $v^{-1} - v$ in the third equation of the system (3.8) by $-\nabla\Psi(v)$, we have

$$\begin{aligned} \overline{\mathcal{A}}(d_x) &= 0, \\ \overline{\mathcal{A}}^T \Delta y + d_s - \overline{Q}(d_x) &= 0, \\ d_x + d_s &= -\nabla\Psi(v). \end{aligned} \tag{3.13}$$

The system (3.13) with the parametric kernel function (2.12) has a unique solution $(d_x, \Delta y, d_s)$ for each $\mu > 0$, which can be used to computer the search directions Δx and Δs from (3.7). If $(x, y, s) \neq (x(\mu), y(\mu), s(\mu))$, then $(\Delta x, \Delta y, \Delta s)$ is nonzero.

By taking a default step size α along the search directions, we get the new iteration point as follows

$$x_+ := x + \alpha\Delta x,\ y_+ := y + \alpha\Delta y,\ \text{and}\ s_+ := s + \alpha\Delta s. \tag{3.14}$$

Furthermore, we can conclude that

$$x \circ s = \mu e \Leftrightarrow v = e \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow \Psi(v) = 0. \tag{3.15}$$

Hence, the value of $\Psi(v)$ can be considered as a measure for the distance between the given iterate (x, y, s) and the corresponding μ-center $(x(\mu), y(\mu), s(\mu))$.

The generic form of primal-dual kernel function-based IPMs for CQSCO is shown in Figure 1.

Generic Primal-Dual Kernel Function-Based IPMs for CQSCO

Input:
 A threshold parameter $\tau \geq 1$;
 an accuracy parameter $\varepsilon > 0$;
 a fixed barrier update parameter $\theta, 0 < \theta < 1$;
 a starting point (x^0, y^0, s^0) with $\mu^0 = \langle x^0, s^0\rangle / r$ such that $\Psi(x^0, s^0; \mu^0) \leq \tau$.
 begin
  $x := x^0$; $y := y^0$; $s := s^0$; $\mu := \mu^0$;
  while $r\mu \geq \varepsilon$ **do**
  begin
   $\mu := (1 - \theta)\mu$;
   while $\Psi(x, s; \mu) > \tau$ **do**
   begin
    calculate the search direction $(\Delta x, \Delta y, \Delta s)$;
    determine the default step size α;
    update $(x, y, s) := (x, y, s) + \alpha(\Delta x, \Delta y, \Delta s)$.
   end
  end
 end

Figure 1. Generic Primal-Dual IPMs for CQSCO

3.2. **Analysis of the kernel function-based IPMs.** From (3.14) and (3.7), after some elementary reductions, we have

$$x_+ = \sqrt{\mu} P(w^{(j)})^{1/2}(v + \alpha d_x),\ \text{and}\ s_+ = \sqrt{\mu} P(w)^{-1/2}(v + \alpha d_s).$$

Furthermore, we have, by Lemma 2.3,

$$v_+ = P(w_+)^{-1/2} P(w)^{1/2}(v + \alpha d_x) = P(w_+)^{1/2} P(w)^{-1/2}(v + \alpha d_s),$$

where

$$w_+ = P(x_+)^{1/2}((P(x_+)^{1/2} s_+)^{-1/2}).$$

To calculate a decrease of the barrier function $\Psi(v)$ during an inner iteration it is standard to consider a decrease as a function of α defined by

$$f(\alpha) := \Psi(v_+) - \Psi(v).$$

Our aim is to find an upper bound for $f(\alpha)$ by using the exponential convexity of $\psi(t)$, and according to Theorem 2.11.

From Lemma 2.4, we have

$$\sqrt{\mu} v_+ = P(w_+)^{\frac{1}{2}} s_+ \sim \left(P(x_+)^{\frac{1}{2}} s_+ \right)^{\frac{1}{2}},$$

and

$$\left(\mu P(v + \alpha d_x)^{\frac{1}{2}} (v + \alpha d_s) \right)^{\frac{1}{2}} \sim \left(\mu P \left(P(w)^{\frac{1}{2}} (v + \alpha d_x) \right)^{\frac{1}{2}} P(w)^{-\frac{1}{2}} (v + \alpha d_s) \right)^{\frac{1}{2}}.$$

Since

$$\left(P(x_+)^{\frac{1}{2}} s_+ \right)^{\frac{1}{2}} = \left(\mu P \left(P(w)^{\frac{1}{2}} (v + \alpha d_x) \right)^{\frac{1}{2}} P(w)^{-\frac{1}{2}} (v + \alpha d_s) \right)^{\frac{1}{2}},$$

we can conclude that v_+ is unitarily similar to $(P(v + \alpha d_x)^{1/2} (v + \alpha d_s))^{1/2}$. This shows that the eigenvalues of v_+ are precisely the same as those of

$$\overline{v}_+ = (P(v + \alpha d_x)^{1/2} (v + \alpha d_s))^{1/2}.$$

From Theorem 2.11, we have

$$\Psi(v_+) = \Psi(\overline{v}_+) \leq \frac{1}{2} \left(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s) \right).$$

Then

$$f(\alpha) \leq f_1(\alpha) := \frac{1}{2} \left(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s) \right) - \Psi(v),$$

which means that $f_1(\alpha)$ gives an upper bound for the decrease of the barrier function $\Psi(v)$. It is worth pointing out that $f_1(\alpha)$ is convex and in general $f(\alpha)$ is not convex. That is an important advantage of using the function $f_1(\alpha)$ instead of using the original decrease function $f(\alpha)$. Moreover, we have $f(0) = f_1(0) = 0$.

We have, by (2.24),

$$f_1'(\alpha) = \frac{1}{2} \left(\mathrm{tr}(\psi'(v + \alpha d_x) \circ d_x) + \mathrm{tr}(\psi'(v + \alpha d_s) \circ d_s) \right).$$

It follows from (3.13) that

$$f_1'(0) = \frac{1}{2} \mathrm{tr}(\nabla \Psi(v) \circ (d_x + d_s)) = -\frac{1}{2} \|\nabla \Psi(v)\|_F^2 = -2\delta(v)^2 < 0.$$

Let

$$d_x = \sum_{i=1}^{r} d_{xi} c_i + \sum_{i<j} d_{xij}$$

be the Peirce decomposition of d_x with respect to the Jordan frame $\{c_1, \ldots, c_r\}$, and

$$d_s = \sum_{i=1}^{r} d_{si} b_i + \sum_{i<j} d_{sij}$$

be the Peirce decomposition of d_s with respect to the Jordan frame $\{b_1, \ldots, b_r\}$. Furthermore, we can write

$$v + \alpha d_x = \sum_{i=1}^{r} \lambda_i(v + \alpha d_x)c_i, \ v + \alpha d_s = \sum_{i=1}^{r} \lambda_i(v + \alpha d_s)b_i.$$

To simplify the notations we used (and will use below), $\eta_i = \lambda_i(v + \alpha d_x)$ and $\gamma_i = \lambda_i(v + \alpha d_s)$ for $i = 1, \cdots, r$.

From (2.25), we have

$$f_1''(\alpha) = g_1(\alpha) + g_2(\alpha), \tag{3.16}$$

where

$$g_1(\alpha) = \sum_{i=1}^{r} \psi''(\lambda_i(\eta))(d_{xi})^2 + \sum_{j<k} \frac{\psi'(\lambda_j(\eta)) - \psi'(\lambda_k(\eta))}{\lambda_j(\eta) - \lambda_k(\eta)} \mathrm{tr}\left((d_{xij})^2\right),$$

and

$$g_2(\alpha) = \sum_{i=1}^{r} \psi''(\lambda_i(\gamma))(d_{si})^2 + \sum_{j<k} \frac{\psi'(\lambda_j(\gamma)) - \psi'(\lambda_k(\gamma))}{\lambda_j(\gamma) - \lambda_k(\gamma)} \mathrm{tr}\left((d_{sij})^2\right).$$

When $\lambda_j(\eta) = \lambda_k(\eta)$ and $\lambda_j(\gamma) = \lambda_k(\gamma)$, the quotients are understood as the second derivatives of $\psi(\lambda_j(\eta))$ and $\psi(\lambda_j(\gamma))$, i.e., $\psi''(\lambda_j(\eta))$ and $\psi''(\lambda_j(\gamma))$, respectively.

It follows directly from (2.26) that

$$\begin{aligned} f_1''(\alpha) \leq & \frac{1}{2}\sum_{i=1}^{r} \psi''(\lambda_i(\eta))(d_{xi})^2 + \sum_{j<k} \psi''(\lambda_k(\eta))\mathrm{tr}\left((d_{xjk})^2\right) \\ & + \frac{1}{2}\sum_{i=1}^{r} \psi''(\lambda_i(\gamma))(d_{si})^2 + \sum_{j<k} \psi''(\lambda_k(\gamma))\mathrm{tr}\left((d_{sjk})^2\right). \end{aligned} \tag{3.17}$$

Note that this makes clear that $f_1''(\alpha) > 0$ 贈測 unless $d_x = d_s = 0$. Thus, since during an inner iteration the iterates x and s are not both at the μ-center, we may conclude that $f_1(\alpha)$ is strictly convex as a function of α.

In what follows, $\delta(v)$ is denoted by δ. The following lemma provides an upper bound of $f_1''(\alpha)$ in terms of δ and $\psi''(t)$.

Lemma 3.1 (Lemma 5.4 in [33]). *One has*

$$f_1''(\alpha) \leq 2\delta^2\psi''(\lambda_{min}(v) - 2\alpha\delta).$$

Recall that $f(\alpha) \leq f_1(\alpha)$ and $f(0) = f_1(0) = -2\delta(v)^2 < 0$. We note that the best value for α is the one that minimizes $f(\alpha)$. The idea underlying our approach is that the step size that minimizes $f(\alpha)$ will be good enough for our purpose. Thus, we want to find α^* such that $f_1'(\alpha^*) = 0$. Since $f_1(\alpha)$ is strictly convex, we have

$$\alpha^* = \max\{\alpha : f_1'(\alpha) \leq 0\}.$$

The default step size that we are going to use will satisfy $f_1'(\alpha) \leq 0$, and as a consequence also $\alpha \leq \alpha^*$. This has as an important consequence that our step size will certainly be feasible.

By integrating we derive from Lemma 3.1 that

$$\begin{aligned} f_1'(\alpha) &= f_1'(0) + \int_0^\alpha f_1''(\xi)d\xi \leq -2\delta^2 + \int_0^\alpha 2\delta^2\psi''(v_{min} - 2\xi\delta)d\xi \\ &= -2\delta^2 - \delta(\psi'(v_{min} - 2\alpha\delta) - \psi'(v_{min})). \end{aligned}$$

Hence, $f_1'(\alpha) \leq 0$ will certainly hold if α satisfies

$$-\psi'(v_{min} - 2\alpha\delta) + \psi'(v_{min}) \leq 2\delta. \tag{3.18}$$

Any α satisfying this inequality will also satisfy $\alpha \leq \alpha^*$, and hence is a feasible step size. Of course, we want α to be as large as possible. Thus our next task is to find the largest α that satisfies (3.18). The detailed can be refer to [4].

Since $f_1(\alpha)$ is strictly convex, we will have $f_1(\alpha) \leq 0$ for all α less than or equal to the value where $f_1(\alpha)$ is minimal and vice versa. Suppose that the step size α satisfies (3.18), then the largest possible value of the step size of α satisfying (3.18) is given by

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)). \tag{3.19}$$

Furthermore, we can conclude that

$$\frac{1}{\psi''(\rho(2\delta))} \leq \bar{\alpha} \leq \frac{1}{\psi''(\rho(\delta))}. \tag{3.20}$$

Then the default step size in the analysis of the kernel function-based IPMs is chosen by

$$\tilde{\alpha} := \frac{1}{\psi''(\rho(2\delta))}. \tag{3.21}$$

From Lemma 4.1 and the fact that $f(\alpha) \leq f_1(\alpha)$, which is a twice differentiable convex function with $f_1(0) = 0$, and $f_1'(0) = -2\delta^2 < 0$, we have the following lemma.

Lemma 3.2. *If step size α satisfies the condition $\alpha \leq \tilde{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

From Lemma 3.2 and (3.21), we have the following theorem, which shows that the default step size (3.21) yields sufficient decrease of the barrier function value during each inner iteration.

Theorem 3.3. *If $\tilde{\alpha}$ is the default step size given by (3.21), then*

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''\left(\rho\left(2\delta\right)\right)}.$$

Let Ψ_0 be the value of $\Psi(v)$ after the μ-update, and Ψ_k, $k = 1, 2, \cdots, K$ be the subsequent values in the same outer iteration, where K is the total number of inner iterations in the outer iteration. Let the constants $\beta > 0$ and $\gamma \in (0, 1]$ be such that for $\Psi(v) \geq \tau$. The definition of K implies $\Psi_{K-1} > \tau$ and $\Psi_K \leq \tau$ and

$$\Psi_{k+1} \leq \Psi_k - \beta(\Psi_k)^{1-\gamma},\ k = 0, 1, \ldots, K - 1.$$

From Lemma 4.2 with $t_k = \Psi_k$, the upper bound of the number of inner iterations is given by

$$K \leq \frac{\Psi_0^\gamma}{\beta\gamma}.$$

The number of outer iterations coincides with the number of barrier parameter θ updates until we obtain $r\mu < \varepsilon$. It is well known (cf. Lemma II.17 in [26]) that the number of outer iterations is bounded above by $\frac{1}{\theta}\log\frac{r}{\varepsilon}$. Thus, an upper bound on the total number of iterations is obtained by multiplying the number of outer iterations and the number of inner iterations.

Theorem 3.4. *The total number of iterations of the algorithm depicted in Fig. 1 is bounded above by*

$$\frac{\Psi_0^\gamma}{\theta\beta\gamma}\log\frac{r}{\varepsilon}.$$

Theorem 3.4 implies that the iteration bound of the algorithm depends on the parameters θ, β, γ and the upper bound on Ψ_0.

3.3. **Complexity of the kernel function-based IPMs.** Following the analysis of kernel function-based IPMs for LO in [4], the iteration bounds for large-update methods can be performed in a systematic way by using the following strategy.

- **Step 1**: Solve the equation $-\frac{1}{2}\psi'(t) = s$ to get $\rho(s)$, the inverse function of $-\frac{1}{2}\psi'(t), t \in (0,1]$. If the equation is hard to solve, derive a lower bound for $\rho(s)$.

Let $-\frac{1}{2}\psi'(t) = s$ for $t \in (0,1]$. Then

$$-t + \frac{1}{t} + \frac{u^2}{2p(t+2u)^2}\tan^{2p}(h(t)) = 2s.$$

This implies that

$$\tan^{2p}(h(t)) = \frac{2p(t+2u)^2}{u^2}\left(2s + t - \frac{1}{t}\right) \leq \frac{4(1+2u)^2 ps}{u^2}.$$

Hence, putting $t = \rho(2\delta)$, we have $4\delta = -\psi'(t)$. Thus

$$\tan^{2p}(h(t)) \leq 8u^{-2}(1+2u)^2 p\delta := \delta_u,$$

This implies that

$$\tan(h(t)) \leq (\delta_u)^{\frac{1}{2p}}. \tag{3.22}$$

Remark: For this special parametric kernel function, we only derive the inequality (3.22) from the equation $-\frac{1}{2}\psi'(t) = s$, not a lower bound for $\rho(s)$. This inequality is enough for the analysis of the algorithms.

- **Step 2**: Calculate the decrease of $\Psi(v)$ in terms of $\delta(v)$ for the default step size $\tilde{\alpha}$ from

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))}.$$

From Lemma 4.4, we have

$$1+\tan(h(t)) > \frac{4u}{3\pi(1+2u)t}, \quad 0<t\le 1,$$

which implies that

$$\frac{1}{t} < \frac{3\pi(1+2u)}{4u}\left(1+\tan(h(t))\right), \quad 0<t\le 1.$$

Note that $\frac{1}{t+2u} < \frac{1}{2u}$ for $0<t\le 1$, together with $p>1$, then we have

$$\begin{aligned}
\tilde{\alpha} &= \frac{1}{\psi''\left(\rho\left(2\delta\right)\right)} \\
&= \left(1+\frac{1}{t^2}+\frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t))+\frac{\pi u^3(1+2u)}{(t+2u)^4}\tan^{2p-1}(h(t))\sec^2(h(t))\right)^{-1} \\
&\ge \left(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+(\delta_u)^{\frac{1}{2p}}\right)^2+\frac{1}{8u}\delta_u+\frac{\pi(1+2u)}{16u}(\delta_u)^{\frac{2p-1}{2p}}\left(1+(\delta_u)^{\frac{2}{2p}}\right)\right)^{-1}.
\end{aligned}$$

Since $\sqrt{2}\delta \ge \sqrt{\Psi(v)} \ge 1$ (see, (3.25)) and $p\in N, p>1$, we can conclude that $p\delta>1$. After some elementary reductions, we have

$$\begin{aligned}
\tilde{\alpha} &\ge \Bigg(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+\left(2\sqrt{2}u^{-1}(1+2u)\right)^{\frac{1}{p}}\right)^2+\frac{(1+2u)^2}{u^3} \\
&\quad +\frac{\pi(1+2u)}{16u}\left(8u^{-2}(1+2u)^2\right)^{\frac{2p-1}{2p}}\left(1+\left(8u^{-2}(1+2u)^2\right)^{\frac{2}{2p}}\right)\Bigg)^{-1}(p\delta)^{-\frac{2p+1}{2p}} \\
&\ge \Bigg(1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+2\sqrt{2}u^{-1}(1+2u)\right)^2+\frac{(1+2u)^2}{u^3} \\
&\quad +\frac{\pi(1+2u)}{16u}\frac{8(1+2u)^2}{u^2}\left(1+8u^{-2}(1+2u)^2\right)\Bigg)^{-1}(p\delta)^{-\frac{2p+1}{2p}} \\
&= \frac{1}{C(u)(p\delta)^{\frac{2p+1}{2p}}},
\end{aligned}$$

where

$$\begin{aligned}
C(u) &= 1+\frac{9\pi^2(1+2u)^2}{16u^2}\left(1+2\sqrt{2}u^{-1}(1+2u)\right)^2 \\
&\quad +\frac{(1+2u)^2}{u^3}+\frac{\pi(1+2u)^3}{2u^3}+\frac{4\pi(1+2u)^5}{u^5}.
\end{aligned}$$

Then

$$f(\tilde{\alpha}) \le -\frac{\delta^2}{C(u)(p\delta)^{\frac{2p+1}{2p}}}. \tag{3.23}$$

- **Step 3**: Solve the equation $\psi(t)=s$ to get $\varrho(s)$, the inverse function of $\psi(t), t\ge 1$. If the equation is hard to solve, derive the lower and upper bounds for $\varrho(s)$.

From (2.21), we have

$$s = \psi(t) = \frac{t^2-1}{2} + \psi_b(t) \leq \frac{t^2-1}{2},\ t \geq 1.$$

The inequality holds due to the fact that $\psi_b(1) = 0$ and $\psi_b(t)$ is monotonically decreasing. Then

$$t = \varrho(s) \geq \sqrt{1+2s}.$$

On the other hand, it follows from Lemma 2.8 that

$$s = \psi(t) \geq \frac{1}{2}(t-1)^2,$$

we have

$$t = \varrho(s) \leq 1 + \sqrt{2s}.$$

Then the lower and upper bounds for $\varrho(s)$ are given by

$$\sqrt{1+2s} \leq \varrho(s) \leq 1 + \sqrt{2s}. \tag{3.24}$$

- **Step 4**: Derive a lower bound for $\delta(v)$ in terms of $\Psi(v)$.

From Lemma 2.8, we have

$$2\delta^2(v) = \frac{1}{2}\sum_{i=1}^{r} \psi'(\lambda_i(v))^2 \geq \sum_{i=1}^{r} \psi(\lambda_i(v)) = \Psi(v).$$

This implies that

$$\delta(v) \geq \sqrt{\frac{\Psi(v)}{2}}. \tag{3.25}$$

- **Step 5**: Using the results of **Step 3** and **Step 4** find positive constants β and γ, with $\gamma \in (0, 1]$, such that

$$f(\tilde{\alpha}) \leq -\beta\Psi(v)^{1-\gamma}.$$

From (3.23) and (3.25), we have

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{C(u)(p\delta)^{\frac{2p+1}{2p}}} \leq -\frac{\delta^{\frac{2p-1}{2p}}}{C(u)p^{\frac{2p+1}{2p}}} \leq -\frac{1}{\sqrt{2}C(u)p^{\frac{2p+1}{2p}}}\Psi(v)^{\frac{2p-1}{4p}}.$$

Thus it follows that $f(\tilde{\alpha}) \leq -\beta\Psi(v)^{1-\gamma}$ for the values of β and γ as given by

$$\beta = \frac{1}{\sqrt{2}C(u)p^{\frac{2p+1}{2p}}},\ \gamma = \frac{2p+1}{4p}. \tag{3.26}$$

- **Step 6**: Calculate the uniform upper bound Ψ_0 for $\Psi(v)$.

Let $\Psi(v) \leq \tau$, we have, by Theorem 2.13,

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2r\tau} + r\right). \tag{3.27}$$

- **Step 7**: Derive an upper bound for the total number of the iterations from

$$\frac{\Psi_0^\gamma}{\theta\beta\gamma}\log\frac{r}{\varepsilon}.$$

Substitution of the expressions in (3.26) and (3.27) yields the following upper bound for the total number of the iterations, namely,

$$O\left(\frac{C(u)p^{\frac{2p+1}{2p}}}{\theta}\left(\tau+\frac{\theta}{2(1-\theta)}\left(2\tau+2\sqrt{2r\tau}+r\right)\right)^{\frac{2p+1}{4p}}\log\frac{r}{\varepsilon}\right). \tag{3.28}$$

- **Step 8**: Set $\tau = O(r)$ and $\theta = \Theta(1)$ so as to calculate an iteration bound for large-update methods.

For large-update methods, we take $\tau = O(r)$ and $\theta = \Theta(1)$. It follows from (3.28) that the iteration bound for large-update methods is given by

$$O\left(p^{\frac{2p+1}{2p}}r^{\frac{2p+1}{4p}}\log\frac{r}{\varepsilon}\right).$$

Let $p = O(\log r)$. Then the iteration bound for large-update methods reduces to

$$O\left(\sqrt{r}\log r\log\frac{r}{\varepsilon}\right),$$

which matches the currently best known iteration bound for large-update methods.

Remark: In case of small-update methods, we have $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{r}})$. It follows from (3.28) that the iteration bound for small-update methods is given by

$$O\left(p^{\frac{2p+1}{2p}}r^{\frac{6p+1}{8p}}\log\frac{r}{\varepsilon}\right).$$

This implies that if the above analysis is used for small-update method, the iteration bound would not be as good as it can be for these types of methods, namely,

$$O\left(\sqrt{r}\log\frac{r}{\varepsilon}\right).$$

This due to the fact that the used upper bound for $\varrho(s)$ given by (3.24) is not tight at $s = 0$. It should be equal to $\varrho(0) = 1$ when $s = 0$. To save space, we leave it as an exercise to the reader to verify that the small-update IPMs based on the proposed parametric kernel function also has the the currently best known iteration bound. For more details, we refer to [4].

4. Conclusions and remarks

In this paper, we have investigated a class of primal-dual large-update IPMs for CQSCO described in Fig. 1 based on the new parametric kernel function with trigonometric barrier term. For this parametric kernel function, we have shown that the best result of iteration bounds for large-update methods can be achieved. In our future study, we intend to generalize the primal-dual IPMs to general nonlinear symmetric cone optimization based on this parametric kernel function.

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Appendix A. Some technical lemmas

Lemma 4.1 (Lemma 12 in [25])**.** *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2}, \quad 0 \leq t \leq t^*.$$

Lemma 4.2 (Lemma 14 in [25])**.** *Suppose $t_0, t_1, \dots, t_K$ is a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma},\ k = 0, 1, \dots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \left\lfloor \frac{t_0^\gamma}{\beta\gamma} \right\rfloor$.

Lemma 4.3. *Let $g(u)$ be as defined in (2.14). Then*

$$g(u) > 0, \quad 0 < u < u^*.$$

Proof. From (2.14) and the fact that $\cos(x) = \sin(\frac{\pi}{2} - x) < \frac{\pi}{2} - x$ for $0 \leq x < \frac{\pi}{2}$, we have

$$\begin{aligned} g'(u) &= -\frac{\pi}{2}\sec^2\left(\frac{\pi(1-2u)}{4}\right) + \frac{4}{3\pi(1+2u)^2} \\ &= \sec^2\left(\frac{\pi(1-2u)}{4}\right)\left(-\frac{\pi}{2} + \frac{4}{3\pi(1+2u)^2}\cos^2\left(\frac{\pi(1-2u)}{4}\right)\right) \end{aligned}$$

$$\begin{aligned}
&\leq \sec^2\left(\frac{\pi(1-2u)}{4}\right)\left(-\frac{\pi}{2}+\frac{4}{3\pi(1+2u)^2}\frac{\pi^2(1+2u)^2}{16}\right)\\
&= -\frac{5\pi}{12}\sec^2\left(\frac{\pi(1-2u)}{4}\right) < 0.
\end{aligned}$$

This implies that $g(u)$ is decreasing in $(0, u^*)$. Due to the fact that $g(u^*) = 0$, we can conclude that $g(u) > 0$ for $0 < u < u^*$. This completes the proof. $\square$

Lemma 4.4. *Let $h(t)$ be as defined in (2.13). Then*

$$f(t,u) := \tan(h(t)) - \frac{4u}{3\pi(1+2u)t} > 0, \qquad 0 < t \leq 2u\ ,\ 0 < u < u^*.$$

Proof. Let $0 < t \leq 1$. Then $0 \leq h(t) < \frac{\pi}{2}$, therefore $\cos(h(t)) \leq \frac{\pi}{2} - h(t)$. Differentiating the function $f(t, u)$ with respect to t, we have

$$\begin{aligned}
\frac{\partial f(t,u)}{\partial t} &= \frac{1}{\cos^2 h(t)}h'(t) + \frac{4u}{3(1+2u)\pi t^2}\\
&= \frac{1}{3\pi t^2 \cos^2 h(t)}\left(3\pi t^2 h'(t) + \frac{4u}{1+2u}\cos^2 h(t)\right)\\
&\leq \frac{1}{3\pi t^2 \cos^2 h(t)}\left(3\pi t^2 h'(t) + \frac{4u}{1+2u}\left(\frac{\pi}{2} - h(t)\right)^2\right)\\
&= \frac{1}{3\pi t^2 \cos^2 h(t)}\left(-3\pi t^2\frac{\pi u(1+2u)}{(t+2u)^2} + \frac{4u}{1+2u}\frac{\pi^2(1+2u)^2 t^2}{4(t+2u)^2}\right)\\
&= -\frac{2\pi u(1+2u)}{3(t+2u)^2\cos^2 h(t)} < 0.
\end{aligned}$$

This implies that $f(t, u)$ is strictly monotonically decreasing with respect to $t \in (0, 2u]$. It follows from Lemma 4.3 that

$$f(2u,u) = \tan\left(\frac{(1-2u)\pi}{4}\right) - \frac{2}{3\pi(1+2u)} = g(u) > 0,\ \ 0 < u < u^*.$$

Then, we can conclude that $f(t, u) > 0$ for $t \in (0, 2u]$. This completes the proof. $\square$

Lemma 4.5 (Lemma 2 in [9])**.** *Let a be a constant, and*

$$w(t,\lambda) = L_n(\lambda)t^n + L_{n-1}(\lambda)t^{n-1} + \cdots + L_1(\lambda)t + L_0(\lambda),\ \ t \in \mathbf{R}.$$

Here $L_i(\lambda)$ are functions of parameter $\lambda \in \mathbf{R}$ for $i = 0, 1, \cdots, n$. If $L_n(\lambda) > 0$, $w(a, \lambda) > 0$ and $\frac{\partial^i w(t,\lambda)}{\partial t^i}|_{t=a} > 0$ for $i = 1, \cdots, n-1$, then we have $w(t, \lambda) > 0$ for all $t > a$.

Appendix B. Proof of lemma 2.7

Proof. We first prove (18-a). The second derivative of $\psi(t)$ is given in (2.16). Using that $\tan(h(t)) > 0$ for all $0 < t < 1$, thus $\psi''(t) > 1$ for $0 < t < 1$.

Now let $t \geq 1$. Define the function

$$\xi(t) := \frac{1}{t^2} + \frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t)) + \frac{\pi u^3(1+2u)}{(t+2u)^4}\tan^{2p-1}(h(t))\sec^2(h(t)),$$

we need to prove that when $0 < u < u^*$ and $t \geq 1$, $\xi(t) > 0$ holds. To do this we consider the following two cases:

Case 18-a.1: For $0 < u \leq \frac{1}{4}$. Then we have $-\frac{\pi}{4} \leq -\pi u < h(t) < 0$. This implies that $-1 < -\tan(\pi u) < \tan(h(t)) \leq 0$ for $t \geq 1$. We have

$$\begin{aligned}\xi(t) &\geq \frac{1}{t^2} + \frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t)) - \frac{2\pi u^3(1+2u)}{(t+2u)^4} \\ &= \frac{u^2}{p(t+2u)^3}\tan^{2p}(h(t)) + \frac{\eta_1(t)}{t^2(t+2u)^4},\end{aligned}$$

where

$$\eta_1(t) := (t+2u)^4 - 2\pi u^3(1+2u)t^2, \ t \geq 1.$$

One can easily verify that

$$\eta_1(1) = (1+2u)^4 - 2\pi u^3(1+2u) = (1+2u)((1+2u)^3 - 2\pi u^3) > 0.$$

Similarly, we can prove that $\eta_1{}'(1) > 0$, $\eta_1{}''(1) > 0$ and $\eta_1{}'''(t) = 24(t+2u) > 0$. From Lemma 4.5, we have $\eta_1(t) > 0$ for $t \geq 1$. This shows that $\psi''(t) > 1$ holds when $0 < u \leq \frac{1}{4}$ and $t > 0$.

Case 18-a.2: Let $\frac{1}{4} < u \leq u^*$. We consider two situations to prove that $\xi(t) > 0$ holds for $t \geq 1$.

Situation 18-a.2.1: Let $1 \leq t < \frac{6u}{4u-1}$. Then $-\frac{\pi}{4} < h(t) \leq 0$, which implies that $-1 < \tan(h(t)) \leq 0$. Similar to the proof in the Case 18-a.1, we can easily verify that (18-a) holds.

Situation 18-a.2.2: Let $t \geq \frac{6u}{4u-1}$. Then $-\pi u^* < h(t) \leq -\frac{\pi}{4}$, which implies that $-\tan(\pi u^*) < \tan(h(t)) \leq -1$. We have

$$\begin{aligned}\xi(t) &= \frac{1}{\tan^{2p}(h(t))}\left(\frac{\tan^{2p}(h(t))}{t^2} + \frac{u^2}{p(t+2u)^3} + \frac{\pi u^3(1+2u)}{(t+2u)^4}\tan^{-1}(h(t))\sec^2(h(t))\right) \\ &\geq \frac{1}{\tan^{2p}(h(t))}\left(\frac{1}{t^2} + \frac{u^2}{p(t+2u)^3} - \frac{\pi u^3(1+2u)\sec^2(\pi u^*)}{(t+2u)^4}\right) \\ &= \frac{1}{\tan^{2p}(h(t))}\left(\frac{u^2}{p(t+2u)^3} + \frac{\eta_2(t)}{t^2(t+2u)^4}\right),\end{aligned}$$

where

$$\eta_2(t) := (t+2u)^4 - \pi u^3(1+2u)\sec^2(\pi u^*)t^2, \ t \geq \frac{6u}{4u-1}.$$

Since $u^* \approx 0.4275 < 0.428$, so $\sec^2(\pi u^*) < \sec^2(0.428\pi) < 20$, and use the fact that $\pi < 3.2$, thus for $\frac{1}{4} < u < u^* < \frac{1}{2}$, we have

$$\begin{aligned}\eta_2\left(\frac{6u}{4u-1}\right) &= \frac{4u^4(1+2u)}{(4u-1)^4}(64(1+2u)^3 - 9\pi u(4u-1)^2)\sec^2(\pi u^*) \\ &> \frac{4u^4(1+2u)}{(4u-1)^4}(64(1+2u)^3 - 576u(4u-1)^2) \\ &= \frac{256u^4(1+2u)}{(4u-1)^4}(1 + 3u(4u-1) + 4u^2 + 68u^2(1-2u))\end{aligned}$$

$$>0.$$

Similarly, we can verify that ${\eta_2}'(\frac{6u}{4u-1}) > 0$, ${\eta_2}''(\frac{6u}{4u-1}) > 0$ and ${\eta_2}'''(t) = 24(t+2u) > 0$. From Lemma 4.5, we have $\eta_2(t) > 0$ when $t \geq \frac{6u}{4u-1}$. This means that $\psi''(t) > 1$ when $\frac{1}{4} < u < u^*$ and $t \geq \frac{6u}{4u-1}$.

From the two cases above we can conclude that (18-a) holds.

By using (2.15) and (2.16), we have

$$\begin{aligned} t\psi''(t) + \psi'(t) =& 2t + \frac{u^2(t-2u)}{2p(t+2u)^3}\tan^{2p}(h(t)) \\ &+ \frac{\pi u^3(1+2u)t}{(t+2u)^4}\tan^{2p-1}(h(t))(1+\tan^2(h(t))). \end{aligned}$$

We will consider three cases to prove (18-b).

Case 18-b.1: Let $0 < t \leq 2u$. After some elementary reductions, we have, by Lemma 4.4,

$$t\psi''(t) + \psi'(t) > 2t + \frac{\pi u^3(1+2u)t}{(t+2u)^4}\tan^{2p-1}(h(t)) + \frac{3u^2t^2 + (8p-12)u^4}{6p(t+2u)^4}\tan^{2p}(h(t)),$$

which indicates that $t\psi'(t) + \psi'(t) > 0$ holds in this case.

Case 18-b.2: Let $2u < t \leq 1$. Then $t - 2u > 0$ and $\tan(h(t)) \geq 0$. It follows that $t\psi''(t) + \psi'(t) > 0$ holds in this case.

Case 18-b.3: Let $t > 1$. It follows from (18-a) that $\psi'(t)$ is an increasing function for $t > 0$. Due to the fact that $\psi'(1) = 0$, we can conclude that $\psi'(t) > 0$ holds for $t > 1$, so $t\psi''(t) + \psi'(t) > 0$.

From the three cases above we can conclude that (18-b) holds.

To prove (18-c), we consider two cases:

Case 18-c.1: Let $0 < u \leq \frac{1}{4}$. Then $-\frac{\pi}{4} \leq -\pi u < h(t) \leq 0$ for $t \geq 1$, which implies that $-1 < \tan(h(t)) \leq 0$ for $t \geq 1$. We have

$$\begin{aligned} t\psi''(t) - \psi'(t) \geq& \frac{2}{t} + \frac{3u^2t + 2u^3}{2p(t+2u)^3}\tan^{2p}(h(t)) - \frac{2\pi u^3(1+2u)t}{(t+2u)^4} \\ =& \frac{3u^2t + 2u^3}{2p(t+2u)^3}\tan^{2p}(h(t)) + \frac{\eta_3(t)}{t(t+2u)^4}, \end{aligned}$$

where

$$\eta_3(t) := 2(t+2u)^4 - 2\pi u^3(1+2u)t^2 = (t+2u)^4 + \eta_1(t) > 0, \ t > 1.$$

Thus $t\psi''(t) - \psi'(t) > 0$ holds for $t > 1$.

Case 18-c.2: Let $\frac{1}{4} < u < u^*$. We consider two situations to prove (18-c) holds in this case.

Situation 18-c.2.1: Let $1 \leq t < \frac{6u}{4u-1}$. Then $-\frac{\pi}{4} < h(t) \leq 0$, which implies that $-1 < \tan(h(t)) \leq 0$. Similar to the proof in the situation 18-a.2.1, we can easily verify that (18-c) holds.

Situation 18-c.2.2: Let $t \geq \frac{6u}{4u-1}$. Then $-\pi u^* < h(t) \leq -\frac{\pi}{4}$, which implies that $-\tan(\pi u^*) < \tan(h(t)) \leq -1$. We have

$$\begin{aligned} t\psi''(t) - \psi'(t) &\geq \frac{1}{\tan^{2p}(h(t))}\left(\frac{2}{t} + \frac{3u^2t + 2u^3}{2p(t+2u)^3} - \frac{\pi u^3(1+2u)\sec^2(\pi u^*)t}{(t+2u)^4}\right) \\ &= \frac{1}{\tan^{2p}(h(t))}\left(\frac{3u^2t + 2u^3}{2p(t+2u)^3} + \frac{\eta_4(t)}{t(t+2u)^4}\right), \end{aligned}$$

where

$$\eta_4(t) := 2(t+2u)^4 - \pi u^3(1+2u)(1+\tan^2(\pi u^*))t^2 = (t+2u)^4 + \eta_2(t) > 0,$$

with $t \geq \frac{6u}{4u-1}$. This implies that $t\psi''(t) - \psi'(t) > 0$ holds when $\frac{1}{4} < u < u^*$ and $t \geq \frac{6u}{4u-1}$.

From the two cases above we can conclude that (18-c) holds.

Finally we need to prove that (18-d) holds.

Using (2.17) and since $\tan(h(t)) > 0$ for $0 < t < 1$, therefore $\psi'''(t) < 0$.

Now let $t \geq 1$. To prove $\psi'''(t) < 0$ we consider two cases.

Case 18-d.1: Let $0 < u \leq \frac{1}{4}$. Then $-\frac{\pi}{4} \leq -\pi u < h(t) \leq 0$ for $t \geq 1$, which implies that $-1 < \tan(h(t)) \leq 0$ for $t \geq 1$. We have

$$\begin{aligned} \psi'''(t) &\leq -\frac{2}{t^3} - \frac{6\pi u^3(1+2u)}{(t+2u)^5}\tan^{2p-1}(h(t))\sec^2(h(t)) \\ &\leq -\frac{2}{t^3} + \frac{12\pi u^3(1+2u)}{(t+2u)^5} = -\frac{2\eta_5(t)}{t^3(t+2u)^5}, \end{aligned}$$

where

$$\eta_5(t) := (t+2u)^5 - 6\pi u^3(1+2u)t^3, \ t \geq 1.$$

Let $0 < u \leq \frac{1}{4}$. Then

$$\eta_5(1) = (1+2u)^5 - 6\pi u^3(1+2u) \geq (1+2u)\left(1 - \frac{3\pi}{32}\right) > 0.$$

Similarly, we can verify that ${\eta_5}'(1) > 0$, ${\eta_5}''(1) > 0$, ${\eta_5}'''(1) > 0$ and ${\eta_5}^{(4)}(t) = 120(t+2u) > 0$. From Lemma 4.5, we have $\eta_5(t) > 0$ for $t \geq 1$. This shows that $\psi'''(t) < 0$ when $0 < u \leq \frac{1}{4}$ and $t \geq 1$.

Case 18-d.2: Let $\frac{1}{4} < u < u^*$. We consider two situations to prove (18-d) holds in this case.

Situation 18-d.2.1: Let $1 \leq t < \frac{6u}{4u-1}$. Then $-\frac{\pi}{4} < h(t) \leq 0$. This implies that $-1 < \tan(h(t)) \leq 0$. Similar to the proof in the case 18-a.2.1, we can easily verify that (18-d) holds.

Situation 18-d.2.2: Let $t \geq \frac{6u}{4u-1}$. Then $-\tan(\pi u^*) < \tan(h(t)) \leq -1$. We have

$$\begin{aligned} \psi'''(t) &\leq -\frac{2}{t^3} - \frac{6\pi u^3(1+2u)}{(t+2u)^5}\tan^{2p-1}(h(t))\sec^2(h(t)) \\ &= -\frac{2}{\tan^{2p}(h(t))}\left(\frac{\tan^{2p}(h(t))}{t^3} + \frac{3\pi u^3(1+2u)}{(t+2u)^5}\tan^{-1}(h(t))\sec^2(h(t))\right) \end{aligned}$$

$$\begin{aligned}&\leq -\frac{2}{\tan^{2p}(h(t))}\left(\frac{1}{t^3}-\frac{3\pi u^3(1+2u)}{(t+2u)^5}\sec^2(\pi u^*)\right)\\&=-\frac{2\eta_6(t)}{t^3(t+2u)^5\tan^{2p}(h(t))},\end{aligned}$$

where

$$\eta_6(t) := (t+2u)^5 - 3\pi u^3(1+2u)\sec^2(\pi u^*)t^3,\ t \geq \frac{6u}{4u-1}.$$

Since $\sec^2(\pi u^*) < 20$ and $\pi < 3.2$, thus for $\frac{1}{4} < u < u^* < \frac{1}{2}$, we have

$$\begin{aligned}\eta_6\left(\frac{6u}{4u-1}\right)&=\left(\frac{4u(1+2u)}{4u-1}\right)^5 - 3\pi(1+\tan^2(\pi u^*))u^3(1+2u)\left(\frac{6u}{4u-1}\right)^3\\&>\frac{512u^5(1+2u)}{(4u-1)^5}\left(2(1+2u)^4-81u(4u-1)^2\right)>0.\end{aligned}$$

Similarly, we can verify that ${\eta_6}'(\frac{6u}{4u-1}) > 0$, ${\eta_6}''(\frac{6u}{4u-1}) > 0$, ${\eta_6}'''(\frac{6u}{4u-1}) > 0$ and ${\eta_6}^{(4)}(t) = 120(t+2u) > 0$. From Lemma 4.5, we have $\eta_6(t) > 0$ when $t \geq \frac{6u}{4u-1}$. This means that $\psi'''(t) < 0$ for $\frac{1}{4} < u < u^*$ and $t \geq \frac{6u}{4u-1}$.

From the above all we complete the proof of the lemma. □

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