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# RECENT PROGRESS IN BILINEAR DECOMPOSITIONS

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Abstract. The targets of this article are twofold. The first one is to give a survey on bilinear decompositions for products of functions in Hardy spaces and their dual spaces, as well as their variants associated with the Schrödinger operator on Euclidean spaces. The second one is to give a new proof of the bilinear decomposition for products of functions in the Hardy space $H^1$ and BMO on metric measure spaces of homogeneous type. Some applications to div-curl lemmas and commutators are also presented.

## 1. Introduction

In this article, we first give a survey on bilinear decompositions for products of functions in Hardy spaces and their dual spaces, as well as their variants associated with the Schrödinger operator, on Euclidean spaces. Then we give a new proof of the bilinear decomposition for products of functions in the Hardy space $H^1$ and BMO on metric measure spaces of homogeneous type. Some applications to div-curl lemmas and commutators as well as some further remarks are also presented.

In what follows, we denote the real Hardy space and the space of functions with bounded mean oscillations on $\mathbb{R}^D$ equipped with the $D$-dimensional Lebesgue measure, respectively, by $H^1(\mathbb{R}^D)$ and $\mathrm{BMO}(\mathbb{R}^D)$. It is well known that the pointwise product $fg$ of $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}(\mathbb{R}^D)$ may not be meaningful, since this pointwise product is not locally integrable on $\mathbb{R}^D$ in general (see [7] for the details). So, three main problems naturally arise:

(i) How do we realize products of functions in $H^1(\mathbb{R}^D)$ and $\mathrm{BMO}(\mathbb{R}^D)$?
(ii) What are the possible decompositions of those products?
(iii) Are there any applications of these decompositions?

For Problem (i), in 2007, Bonami et al. [7] viewed the product $f \times g$ where $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}(\mathbb{R}^D)$ as a *Schwartz distribution*, which is denoted by $f \times g \in \mathscr{S}'(\mathbb{R}^D)$. More precisely, let $\mathscr{S}(\mathbb{R}^D)$ be the *Schwartz class*, equipped with the well-known topology induced by a series of semi-norms, and $\mathscr{S}'(\mathbb{R}^D)$ the dual space of $\mathscr{S}(\mathbb{R}^D)$, equipped with the weak-$*$ topology. We regard $\varphi \in \mathscr{S}(\mathbb{R}^D)$ as a multiplier for $\mathrm{BMO}(\mathbb{R}^D)$, namely, for any $g \in \mathrm{BMO}(\mathbb{R}^D)$, $\varphi g \in \mathrm{BMO}(\mathbb{R}^D)$

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(see [43, 59, 60] for the details). Then $f \times g$ is defined by the rule

$$\langle f \times g, \varphi \rangle := \int_{\mathbb{R}^D} f(x)[\varphi(x)g(x)]\, d\mu(x).$$

The above integral makes sense via the duality between $H^1(\mathbb{R}^D)$ and $\mathrm{BMO}\,(\mathbb{R}^D)$; see the seminal work of Fefferman and Stein [23]. Then Bonami et al. [7] successfully illustrated the meaning of $f \times g$. In what follows, $\mathscr{C}^\infty_{c,0}(\mathbb{R}^D)$ denotes the *space of all smooth functions with compact supports and the vanishing moment of order zero.*

**Lemma 1.1** ([7]). *Let $g \in \mathrm{BMO}\,(\mathbb{R}^D)$. Then the mapping $f \mapsto gf$, which is a priori defined on $\mathscr{C}^\infty_{c,0}(\mathbb{R}^D)$ and takes values in $\mathscr{S}'(\mathbb{R}^D)$, can be extended continuously into a mapping from $H^1(\mathbb{R}^D)$ into $\mathscr{S}'(\mathbb{R}^D)$, which is denoted by $f \mapsto f \times g$. Moreover, for $\{g_k\}_{k\in\mathbb{N}} \subset L^\infty(\mathcal{X})$ with $\lim_{k\to\infty} g_k = g$ almost everywhere on $\mathbb{R}^D$, the sequence $\{f \times g_k\}_{k\in\mathbb{N}}$ converges to $f \times g$ in $\mathscr{S}'(\mathbb{R}^D)$.*

The investigation of the distribution $f \times g$ was motivated by the recent developments in geometric function theory [1, 41, 42] and nonlinear elasticity [3, 58, 70]; see [7] for the details.

On Problems (ii) and (iii), Bonami et al. [7] showed that the product $f \times g$ of $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}\,(\mathbb{R}^D)$ can be further written as a sum of an integrable function and a distribution in some adapted *Hardy-Orlicz space* $H^\Phi(\mathbb{R}^D, \mu)$, where

$$\Phi(t) := t/\log(e+t), \quad \forall\, t \in [0,\infty), \tag{1.1}$$

and $d\mu(x) := dx/\log(e+|x|)$ for all $x \in \mathbb{R}^D$. Let

$$\mathrm{BMO}^+(\mathbb{R}^D) := \Big\{ g \in \mathrm{BMO}\,(\mathbb{R}^D) : \|g\|_{\mathrm{BMO}^+(\mathbb{R}^D)} := \|g\|_{\mathrm{BMO}\,(\mathbb{R}^D)} + \|g\|_{L^1(\mathbb{R}^D)} < \infty \Big\}. \tag{1.2}$$

**Theorem 1.2** ([7]). *For any given $f \in H^1(\mathbb{R}^D)$, there exist two bounded linear operators: $\mathscr{L}_f$ from $\mathrm{BMO}\,(\mathbb{R}^D)$ into $L^1(\mathbb{R}^D)$, and $\mathscr{H}_f$ from $\mathrm{BMO}\,(\mathbb{R}^D)$ into $H^\Phi(\mathbb{R}^D, \mu)$, and a positive constant $C$ such that, for all $g \in \mathrm{BMO}\,(\mathbb{R}^D)$,*

$$f \times g = \mathscr{L}_f g + \mathscr{H}_f g$$

*and*

$$\|\mathscr{L}_f g\|_{L^1(\mathbb{R}^D)} + \|\mathscr{H}_f g\|_{H^1(\mathbb{R}^D)} \le C\|f\|_{H^1(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}^+(\mathbb{R}^D)}.$$

Let $\{\phi_\epsilon\}_{\epsilon\in(0,\infty)}$ be an approximation to the identity and, for any $\epsilon \in (0,\infty)$ and suitable function $f$, $f_\epsilon := f * \phi_\epsilon$. As a consequence of Theorem 1.2, it was shown in [7] that the pointwise product $fg$ of $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}\,(\mathbb{R}^D)$ can be approximated by the convolutions $\{(f \times g)_\epsilon\}_{\epsilon\in(0,\infty)}$ of the distribution $f \times g$.

**Theorem 1.3** ([7]). *Let $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}\,(\mathbb{R}^D)$. Then, for almost every $x \in \mathbb{R}^D$,*

$$f(x)g(x) = \lim_{\epsilon\to 0}(f \times g)_\epsilon(x).$$

**Remark 1.4.** It was shown in [7, Corollary 1.9] that, if the pointwise product $fg$ of $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}\,(\mathbb{R}^D)$ is locally integrable on $\mathbb{R}^D$, then $fg$, as a distribution, coincides with $f \times g \in \mathscr{S}'(\mathbb{R}^D)$.

Theorem 1.2 was further extended by Bonami and Feuto [4] to the classical Hardy space $H^p(\mathbb{R}^D)$ with $p \in (0,1)$. Let $\dot{\Lambda}_{D(\frac{1}{p}-1)}(\mathbb{R}^D)$ and $\Lambda_{D(\frac{1}{p}-1)}(\mathbb{R}^D)$ be the *homogeneous* and the *inhomogeneous Lipschitz spaces* respectively. For each $\alpha \in (D(1-p),\infty)$, $H^p_{w_\alpha}(\mathbb{R}^D)$ is the *weighted Hardy space* with the weight function

$$w_\alpha(x) := \frac{1}{(1+|x|)^\alpha}, \quad \forall x \in \mathbb{R}^D.$$

For any $p \in (\frac{D}{D+1},1)$, Bonami and Feuto [4] obtained the following linear decompositions:

$$H^p(\mathbb{R}^D) \times \Lambda_{D(\frac{1}{p}-1)}(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + H^p(\mathbb{R}^D) \tag{1.3}$$

and

$$H^p(\mathbb{R}^D) \times \dot{\Lambda}_{D(\frac{1}{p}-1)}(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + H^p_{w_\alpha}(\mathbb{R}^D). \tag{1.4}$$

Theorem 1.3 has some applications in nonlinear PDEs, where $f \times g \in \mathscr{S}'(\mathbb{R}^D)$ is used to justify the weak continuity properties of the pointwise product $fg$.

Bonami et al. [7] further conjectured whether or not the operators $\mathscr{L}_f$ and $\mathscr{H}_f$ in Theorem 1.2 can depend linearly on $f$.

**Conjecture 1.5** ([7])**.** Prove that there exist two bounded bilinear operators:

$$\mathscr{L}: \ H^1(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D) \to L^1(\mathbb{R}^D),$$
$$\mathscr{H}: \ H^1(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D) \to H^\Phi(\mathbb{R}^D,\mu),$$

and a positive constant $C$ such that, for all $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}\,(\mathbb{R}^D)$,

$$f \times g = \mathscr{L}(f,g) + \mathscr{H}(f,g)$$

and

$$\|\mathscr{L}(f,g)\|_{L^1(\mathbb{R}^D)} + \|\mathscr{H}(f,g)\|_{H^\Phi(\mathbb{R}^D,\mu)} \le C\|f\|_{H^1(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}^+(\mathbb{R}^D)}.$$

Recently, using wavelet tools and bilinear estimates of paraproducts, Bonami et al. [6] affirmatively confirmed the above conjecture in a sharp manner, where the aforementioned Hardy-Orlicz space $H^\Phi(\mathbb{R}^D,\mu)$ can be replaced by a smaller space $H^{\log}(\mathbb{R}^D)$, which is a special case of Musielak-Orlicz-type Hardy spaces originally introduced by Ky [47]. Bonami et al. [6] further showed that $H^{\log}(\mathbb{R}^D)$ is optimal in the sense that it can not be replaced by a smaller space by using the main theorem of Nakai and Yabuta [59]; see [6, 47] for more details. For more properties on Musielak-Orlicz-type Hardy spaces, we refer the reader to [37, 47, 53–56, 64] (see also the monograph [63] for a complete theory of Musielak-Orlicz-Hardy spaces).

To be precise, Bonami et al. [6] showed that $f \times g$ can be decomposed into a sum of two bilinear bounded operators, respectively, from $H^1(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D)$ into $L^1(\mathbb{R}^D)$ and from $H^1(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D)$ into $H^{\log}(\mathbb{R}^D)$. As a consequence, they obtained an optimal endpoint estimate involving the space $H^{\log}(\mathbb{R}^D)$ for the div-curl lemma related to an implicit conjecture from [7] (see also [5,6]). Moreover, the above decomposition of the products plays essential roles in establishing the bilinear or the subbilinear decompositions, respectively, for the linear or the sublinear commutators of singular integrals by Ky [46]; see [48, 50] for more applications of the above decompositions.

On the other hand, for local cases, let $h^p(\mathbb{R}^D)$ with $p \in (0,1]$ be the local Hardy space in the sense of Goldberg [30], $\mathrm{bmo}\,(\mathbb{R}^D)$ the local BMO space and $h_*^\Phi(\mathbb{R}^D)$ a variant of the local Orlicz-Hardy space associated to $\Phi$ as in (1.1), which was introduced in [4]. Bonami et al. [4] obtained linear decompositions of the products of the local Hardy spaces and their dual spaces. Precisely, Bonami et al. [4] showed that

$$h^1(\mathbb{R}^D) \times \mathrm{bmo}\,(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + h_*^\Phi(\mathbb{R}^D) \tag{1.5}$$

and, for any $p \in (0,1)$,

$$h^p(\mathbb{R}^D) \times \Lambda_{D(\frac{1}{p}-1)}(\mathbb{R}^D) \subset L^1(\mathbb{R}^D) + h^p(\mathbb{R}^D), \tag{1.6}$$

where both the decompositions in (1.5) and (1.6) are linear only with respect to the functions from $\mathrm{bmo}\,(\mathbb{R}^D)$ or from $\Lambda_\alpha(\mathbb{R}^D)$.

Later, Cao et al. [8] improved the above results in [4, 6, 7] by investigating the bilinear decompositions of the products of local Hardy spaces $h^p(\mathbb{R}^D)$ and their dual spaces in the case when $p < 1$ and near to 1.

By the celebrating work of Fefferman and Stein [23], it is well known that the Hardy space $H^1(\mathbb{R}^D)$ is essentially associated to the Laplace operator $\Delta$. In the past two decades, many researchers turned their attentions to Hardy spaces associated to operators other than $\Delta$ over various settings; see, for example, [20–22, 62, 67]. To be precise, Shen [62] triggered the study of harmonic analysis associated to Schrödinger operators. Dziubański and Zienkiewicz [22] established the characterizations of the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^D)$ associated to the Schrödinger operator $\mathcal{L}$ via atoms, the maximal function defined by the semigroup generated by $\mathcal{L}$ and the Riesz transforms $\nabla\mathcal{L}^{-1/2}$. Dziubański et al. [21] proved that the dual space of $H^1_{\mathcal{L}}(\mathbb{R}^D)$ is the BMO type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$ associated to $\mathcal{L}$, and presented some applications. Later, Duong and Yan [18] obtained the molecular and the Lusin-area function characterizations of Hardy spaces $H^1_{\mathcal{L}}(\mathbb{R}^D)$ associated to $\mathcal{L}$ with heat kernel bounds, including the Schrödinger operator with non-negative potential as a special case. The maximal function characterization, via the semigroup generated by $\mathcal{L}$, and the atomic characterization for the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^D)$ associated to the degenerate Schrödinger operator $\mathcal{L}$ were established by Dziubański [20].

For the product $f \times g$ of $f \in H^1_{\mathcal{L}}(\mathbb{R}^D)$ and $g \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$, Li and Peng [52] proved that $f \times g$, regarded as a distribution, can be written into a sum of two parts: one lies in $L^1(\mathbb{R}^D)$ and the other belongs to some weighted Hardy-Orlicz space associated to $\mathcal{L}$. Ky [48] essentially improved the above result by proving that the product of two functions $f \in H^1_{\mathcal{L}}(\mathbb{R}^D)$ and $g \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$ can be written into a sum of two bilinear operators, which map boundedly from $H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$ to $L^1(\mathbb{R}^D)$, and to the optimal Hardy-Orlicz space $H^{\log}(\mathbb{R}^D)$, respectively. This bilinear decomposition also motivated the study of the endpoint boundedness of commutators of singular integrals associated to $\mathcal{L}$ in [50].

We know that many classical results of harmonic analysis on Euclidean spaces can be extended naturally to spaces of homogeneous type in the sense of Coifman and Weiss [13, 14], or on the RD-space that was introduced by Han, Müller and Yang [36].

Recall that a *space of homogeneous type*, $(\mathcal{X}, d, \mu)$, in the sense of Coifman and Weiss [13,14] is a quasi-metric space $(\mathcal{X}, d)$ equipped with a non-negative measure $\mu$ satisfying the following *measure doubling condition*: there exists a universal constant $C_{(\mathcal{X})} \in [1, \infty)$ such that, for all balls $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ with $(x, r) \in \mathcal{X} \times (0, \infty)$,

$$\mu(B(x, 2r)) \le C_{(\mathcal{X})}\mu(B(x, r)).$$

Equivalently, there exists a positive constant $\widetilde{C}_{(\mathcal{X})}$ such that, for any $\lambda \in [1, \infty)$,

$$\mu(B(x, \lambda r)) \le \widetilde{C}_{(\mathcal{X})}\lambda^n \mu(B(x, r)) \tag{1.7}$$

with $n := \log_2 C_{(\mathcal{X})}$. Let

$$n_0 := \inf\{n \in (0, \infty) : \ n \text{ satisfies (1.7)}\}. \tag{1.8}$$

Notice that $n_0$ can be regarded as the dimension of $\mathcal{X}$, $n_0 \le n$ and (1.7) may not be true if $n$ is replaced by $n_0$.

Recall that a *metric measure space of homogeneous type*, $(\mathcal{X}, d, \mu)$, is a space of homogeneous type with $d$ being a metric and, moreover, an RD-*space* $(\mathcal{X}, d, \mu)$ is a space of homogeneous type, which satisfies the following additional *reverse doubling condition* (see [35, 36]): there exist positive constants $a_0$, $\widehat{C}_{(\mathcal{X})} \in (1, \infty)$ such that, for all balls $B(x, r)$ with $x \in \mathcal{X}$ and $r \in (0, \operatorname{diam}(\mathcal{X})/a_0)$,

$$\mu(B(x, a_0 r)) \ge \widehat{C}_{(\mathcal{X})}\mu(B(x, r))$$

(see [66] for some equivalent characterizations of RD-spaces), here and hereafter,

$$\operatorname{diam}(\mathcal{X}) := \sup\{d(x, y) : \ x, y \in \mathcal{X}\}.$$

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type. Coifman and Weiss [14] introduced the atomic Hardy space $H^{p,q}_{\rm at}(\mathcal{X})$ for all $p \in (0, 1]$ and $q \in [1, \infty] \cap (p, \infty]$ and proved that $H^{p,q}_{\rm at}(\mathcal{X})$ is independent of the choice of $q$, which is simply denoted by $H^p_{\rm at}(\mathcal{X})$ hereafter, and that its dual space is the Lipschitz space $\mathrm{Lip}_{1/p-1}(\mathcal{X})$ when $p \in (0, 1)$, or the space $\mathrm{BMO}(\mathcal{X})$ when $p = 1$.

Moreover, under an additional assumption that there exists a specific generalized approximation of the identity, Duong and Yan [19] developed a theory of new BMO-type function spaces on spaces of homogeneous type.

On any RD-space $(\mathcal{X}, d, \mu)$ with $d$ being a metric, for $p \in (\frac{n_0}{n_0+1}, 1]$ with $n_0$ as in (1.8), Han et al. [35] established a Littlewood-Paley theory for atomic Hardy spaces $H^p_{\rm at}(\mathcal{X})$; Grafakos et al. [33] obtained their characterizations via various maximal functions. Moreover, it was proved in [36] that these Hardy spaces identified with some special cases of Triebel-Lizorkin spaces on $(\mathcal{X}, d, \mu)$. In order to develop a real-variable theory of Hardy spaces or, more generally, Besov spaces and Triebel-Lizorkin spaces on RD-spaces, some basic tools, including spaces of test functions, approximations of the identity and various Calderón reproducing formulae on RD-spaces were well developed in [35, 36]. From then on, these basic tools play crucial roles in harmonic analysis on RD-spaces; see, for example, [32, 34–36, 44, 45, 65, 66].

Let $(\mathcal{X}, d, \mu)$ be an RD-space. The problem about the product of $f \in H^1_{\rm at}(\mathcal{X})$ and $g \in \mathrm{BMO}(\mathcal{X})$ was first studied by Feuto [25]. In [25], Feuto showed that the product of $f \in H^1_{\rm at}(\mathcal{X})$ and $g \in \mathrm{BMO}(\mathcal{X})$, viewed as a distribution, can be written as a sum of an integrable function and a distribution in some adapted Hardy-Orlicz space $H^\Phi(\mathcal{X},\nu)$ with $\Phi$ as in (1.1).

**Theorem 1.6** ([25])**.** *Let $(\mathcal{X},d,\mu)$ be an RD-space. Then, for any $f\in H^1_{\rm at}(\mathcal{X})$ and $g\in \mathrm{BMO}\,(\mathcal{X})$, the product $f\times g$ can be given a meaning in the sense of distributions. Moveover, there exist two bounded linear operators: $\mathscr{L}_f$ from $\mathrm{BMO}\,(\mathcal{X})$ into $L^1(\mathcal{X})$ and $\mathscr{H}_f$ from $\mathrm{BMO}\,(\mathcal{X})$ into $H^\Phi(\mathcal{X},\nu)$, and a positive constant $C$ such that, for all $g\in \mathrm{BMO}\,(\mathcal{X})$,*

$$f\times g=\mathscr{L}_f g+\mathscr{H}_f g$$

*and*

$$\|\mathscr{L}_f g\|_{L^1(\mathcal{X})}+\|\mathscr{H}_f g\|_{H^\Phi(\mathcal{X},\nu)}\le C\|f\|_{H^1_{\rm at}(\mathcal{X})}\|g\|_{\mathrm{BMO}^+(\mathcal{X})},$$

*where $\Phi$ is as in (1.1), $d\nu(x):=\frac{d\mu(x)}{\log(e+d(x_1,x))}$ for all $x\in\mathcal{X}$ and $x_1$ is a fixed point of $\mathcal{X}$.*

Recently, Ky [49] improved the above result via showing that the product $f\times g$ can be written into a sum of two linear operators and via replacing the Hardy-Orlicz space $H^\Phi(\mathcal{X},\nu)$ by some Musielak-Orlicz-type Hardy space $H^{\log}(\mathcal{X})$ which is a subspace of the above Hardy-Orlicz space and is known to be optimal even on Euclidean spaces.

**Theorem 1.7** ([49])**.** *Let $(\mathcal{X},d,\mu)$ be an RD-space. Then, for every $f\in H^1_{\rm at}(\mathcal{X})$, there exist two bounded linear operators: $\mathscr{L}_f$ from $\mathrm{BMO}\,(\mathcal{X})$ into $L^1(\mathcal{X})$ and $\mathscr{H}_f$ from $\mathrm{BMO}\,(\mathcal{X})$ into $H^{\log}(\mathcal{X})$, and a positive constant $C$ such that, for all $g\in \mathrm{BMO}\,(\mathcal{X})$,*

$$f\times g=\mathscr{L}_f g+\mathscr{H}_f g$$

*and*

$$\|\mathscr{L}_f g\|_{L^1(\mathcal{X})}+\|\mathscr{H}_f g\|_{H^{\log}(\mathcal{X})}\le C\|f\|_{H^1_{\rm at}(\mathcal{X})}\|g\|_{\mathrm{BMO}^+(\mathcal{X})}.$$

A. Bonami and F. Bernicot further *conjectured* that $f\times g$ can be written into a sum of two *bilinear* operators, which was presented by Ky in [49, p. 809, Conjecture].

**Conjecture 1.8** ([49])**.** Let $(\mathcal{X},d,\mu)$ be an RD-space. Prove that there exist two bounded bilinear operators:

$$\mathscr{L}:\ H^1_{\rm at}(\mathcal{X})\times \mathrm{BMO}\,(\mathcal{X})\to L^1(\mathcal{X}),\quad \mathscr{H}:\ H^1_{\rm at}(\mathcal{X})\times \mathrm{BMO}\,(\mathcal{X})\to H^{\log}(\mathcal{X})$$

and a positive constant $C$ such that, for all $f\in H^1_{\rm at}(\mathcal{X})$ and $g\in \mathrm{BMO}\,(\mathcal{X})$,

$$f\times g=\mathscr{L}(f,g)+\mathscr{H}(f,g)$$

and

$$\|\mathscr{L}(f,g)\|_{L^1(\mathcal{X})}+\|\mathscr{H}(f,g)\|_{H^{\log}(\mathcal{X})}\le C\|f\|_{H^1_{\rm at}(\mathcal{X})}\|g\|_{\mathrm{BMO}^+(\mathcal{X})}.$$

Recently, Auscher and Hytönen [2] built an orthonormal basis of Hölder continuous wavelets with exponential decay on spaces of homogeneous type, which paved the way for one to confirm Conjecture 1.8. Fu and Yang [26] obtained an unconditional basis of $H^1_{\rm at}(\mathcal{X})$ and several equivalent characterizations of $H^1_{\rm at}(\mathcal{X})$ in terms of wavelets. Fu et al. [27] further proved the bilinear decompositions:

$$H^1_{\rm at}(\mathcal{X}) \times {\rm BMO}\,(\mathcal{X}) \subset L^1(\mathcal{X}) + H^{\log}(\mathcal{X}),$$

where $H^{\log}(\mathcal{X})$ is a space of Musielak-Orlicz-type, which affirmatively confirms Conjecture 1.8. These bilinear decompositions stimulated the investigation of bilinear decompositions of commutators in [57].

Later, under an additional assumption that there exists a specific generalized approximation of the identity (see Assumption 4.14 below), Fu and Yang [28] established a local version of [27, Theorem 1.7] on RD-spaces.

This article is organized as follows.

In Section 2, we summarise the bilinear decompositions for products of functions in $H^1(\mathbb{R}^D)$ and ${\rm BMO}\,(\mathbb{R}^D)$ and their applications on Euclidean spaces. This section is divided into four parts. In Subsections 2.1 and 2.2, we review the bilinear decompositions of products of functions in $H^1(\mathbb{R}^D)$ and ${\rm BMO}\,(\mathbb{R}^D)$, which confirms Conjecture 1.5, and their applications to div-curl lemmas and the bilinear and the subbilinear decompositions, and the endpoint boundedness of commutators. Subsections 2.3 and 2.4 are devoted to reviewing the bilinear decompositions of products of functions in $H^1_{\mathcal{L}}(\mathbb{R}^D)$ and ${\rm BMO}_{\mathcal{L}}(\mathbb{R}^D)$ associated to the Schrödinger operator $\mathcal{L}$, and their applications to the bilinear and the subbilinear decompositions, and the endpoint boundedness of commutators associated to the Schrödinger operator $\mathcal{L}$.

Section 3 aims to recall the bilinear decompositions for products of functions in $h^p(\mathbb{R}^D)$ and Lipschitz spaces $\Lambda_{D(1/p-1)}(\mathbb{R}^D)$ and their applications to the div-curl lemmas.

Section 4 is divided into two parts. In the first part, we give a survey of bilinear decompositions of products of functions in $H^1_{\rm at}(\mathcal{X})$ and ${\rm BMO}\,(\mathcal{X})$ and their applications on spaces of homogeneous type. This part is further decomposed into three subsections. In Subsection 4.1, we review the bilinear decompositions of products of functions in $H^1_{\rm at}(\mathcal{X})$ and ${\rm BMO}\,(\mathcal{X})$. In Subsection 4.2, we summarize the bilinear decompositions of products of functions in $H^1_{\rho}(\mathcal{X})$ and ${\rm BMO}_{\rho}(\mathcal{X})$ associated to the admissible function $\rho$. Subsection 4.3 is devoted to reviewing the applications of the bilinear decompositions in Subsection 4.1 to the endpoint boundedness of commutators. Observe that there exists a gap in the proof of [27, Theorem 1.7] (see Theorem 4.9 below), which can be sealed with some minor modifications that will be presented elsewhere. In Subsection 4.4, we give a new proof of Theorem 4.9.

Some further remarks, including some open questions on spaces of homogeneous type, or even on Euclidean spaces, are presented in Section 5.

## 2. Products of functions in $H^1(\mathbb{R}^D)$ and ${\rm BMO}\,(\mathbb{R}^D)$

In this section, we review the bilinear decompositions for products of functions in $H^1(\mathbb{R}^D)$ and ${\rm BMO}\,(\mathbb{R}^D)$ and their applications.

### 2.1. **Bilinear decompositions for products of functions in $H^1(\mathbb{R}^D)$ and ${\rm BMO}\,(\mathbb{R}^D)$.**

In this subsection, we introduce bilinear decompositions for products of functions in $H^1(\mathbb{R}^D)$ and BMO $(\mathbb{R}^D)$ and their applications to the endpoint estimate involving the space $H^{\log}(\mathbb{R}^D)$ for the div-curl lemma.

We first recall the notions of Hardy spaces from [23]. For every $m \in \mathbb{N}$, $f \in \mathscr{S}'(\mathbb{R}^D)$ and $x \in \mathbb{R}^D$, let

$$f_m^*(x) := \sup_{\varphi \in \mathscr{S}_m(\mathbb{R}^D)} \sup_{\substack{|y-x|<t \\ t\in(0,\infty)}} |f * \varphi_t(y)|,$$

here and hereafter

$$\mathscr{S}_m(\mathbb{R}^D) := \left\{ \varphi \in \mathscr{S}(\mathbb{R}^D) : \sup_{\substack{x\in\mathbb{R}^D \\ |\alpha|\le m+1}} \left[ (1+|x|)^{(m+2)(n+1)} |\partial_x^\alpha \varphi(x)| \right] \le 1 \right\}. \tag{2.1}$$

The *Hardy space* $H^p(\mathbb{R}^D)$ is defined by setting

$$H^p(\mathbb{R}^D) := \left\{ f \in \mathscr{S}'(\mathbb{R}^D) : \|f\|_{H^p(\mathbb{R}^D)} := \|f_m^*\|_{L^p(\mathbb{R}^D)} < \infty \right\},$$

here and hereafter, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^D$, $\varphi_t(x) := \frac{1}{t^n}\varphi(\frac{x}{t})$ and the subscript $m > \lfloor D(1/p-1) \rfloor$ is always omitted. Recall that, for any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the biggest integer which is not bigger than $s$.

For the Hardy space $H^p(\mathbb{R}^D)$, one of its most important properties is its atomic characterization, which was first established by Coifman [11] for $D = 1$ and extended by Latter [51] to $D > 1$.

**Definition 2.1.** Let $p \in (0,1]$, $q \in [1,\infty) \cap (p,\infty]$ and $Q$ be a cube in $\mathbb{R}^D$. A function $a \in L^q(\mathbb{R}^D)$ is called a $(p,q)$-*atom* related to $Q$ if

(i) supp $(a) \subset Q$;
(ii) $\|a\|_{L^q(\mathbb{R}^D)} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$;
(iii) if $|Q| < 1$, then $\int_{\mathbb{R}^D} x^\alpha a(x)\,dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le \lfloor D(\frac{1}{p}-1) \rfloor$.

The following result establishes the atomic characterization of the Hardy space $H^p(\mathbb{R}^D)$ for any $p \in (0,\,1]$.

**Theorem 2.2** ([11, 51])**.** *Let $p \in (0,\,1]$, $q \in [1,\infty) \cap (p,\infty]$ and $f \in H^p(\mathbb{R}^D)$. Then there exist a family $\{a_j\}_{j=1}^\infty$ of $(p,q)$-atoms and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathscr{S}'(\mathbb{R}^D)$. Moreover, there exists a positive constant $C$, independent of $f$, such that*

$$\frac{1}{C}\|f\|_{H^p(\mathbb{R}^D)} \le \left[ \sum_{j=1}^\infty |\lambda_j|^p \right]^{\frac{1}{p}} \le C\|f\|_{H^p(\mathbb{R}^D)}.$$

The Musielak-Orlicz function $\theta$ is defined by setting, for all $x \in \mathbb{R}^D$ and $t \in (0,\infty)$,

$$\theta(x,t) := \frac{t}{\log(e+|x|) + \log(e+t)}.$$

The *Musielak-Orlicz space* $L^{\log}(\mathbb{R}^D)$ is defined as the space of all measurable functions $f$ such that

$$\int_{\mathbb{R}^D} \theta(x,|f(x)|)\,dx < \infty$$

and the *quasi-norm* of $f$ in $L^{\log}(\mathbb{R}^D)$ is defined by setting

$$\|f\|_{L^{\log}(\mathbb{R}^D)} := \inf\left\{\lambda\in(0,\infty):\ \int_{\mathbb{R}^D}\theta\left(x,\frac{|f(x)|}{\lambda}\right)dx\le 1\right\}.$$

Then we recall the notions of the Hardy-type space $H^{\log}(\mathbb{R}^D)$. We first recall the grand maximal function of a distribution $f\in\mathscr{S}'(\mathbb{R}^D)$ as follows. Let $\mathcal{F}$ be the class of all functions $\phi$ in $\mathscr{S}(\mathbb{R}^D)$ satisfying

$$|\phi(x)|+|\nabla\phi(x)|\le(1+|x|)^{-(D+1)}.$$

For any $t\in(0,\infty)$, let $\phi_t(x):=t^{-D}\phi(x/t)$ for all $x\in\mathbb{R}^D$. Then

$$\mathcal{M}f(x):=\sup_{\phi\in\mathcal{F}}\sup_{t\in(0,\infty)}|f*\phi_t(x)|. \tag{2.2}$$

The *Musielak-Orlicz Hardy space* $H^{\log}(\mathbb{R}^D)$ is defined by setting

$$H^{\log}(\mathbb{R}^D):=\left\{f\in\mathscr{S}'(\mathbb{R}^D):\ \|f\|_{H^{\log}(\mathbb{R}^D)}:=\|\mathcal{M}f\|_{L^{\log}(\mathbb{R}^D)}<\infty\right\}.$$

We first state the result in [6, Theorem 1.1], which confirms Conjecture 1.5. Recall that $\mathrm{BMO}^+(\mathbb{R}^D)$ is defined as in (1.2).

**Theorem 2.3** ([6])**.** *There exist two bounded bilinear operators*

$$\mathscr{L}:\ H^1(\mathbb{R}^D)\times\mathrm{BMO}\,(\mathbb{R}^D)\to L^1(\mathbb{R}^D)$$

*and*

$$\mathscr{H}:\ H^1(\mathbb{R}^D)\times\mathrm{BMO}\,(\mathbb{R}^D)\to H^{\log}(\mathbb{R}^D),$$

*and a positive constant $C$ such that, for all $f\in H^1(\mathbb{R}^D)$ and $g\in\mathrm{BMO}\,(\mathbb{R}^D)$,*

$$f\times g=\mathscr{L}(f,g)+\mathscr{H}(f,g)\quad in\quad \mathscr{S}'(\mathbb{R}^D)$$

*and*

$$\|\mathscr{L}(f,g)\|_{L^1(\mathbb{R}^D)}+\|\mathscr{H}(f,g)\|_{H^{\log}(\mathbb{R}^D)}\le C\|f\|_{H^1(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}^+(\mathbb{R}^D)}.$$

The following result from [6, Theorem 1.2] gives an optimal endpoint estimate involving the space $H^{\log}(\mathbb{R}^D)$ for the div-curl lemma, which sharpens a known result in [5, Theorem 1.2]. Let

$$H^1(\mathbb{R}^D;\mathbb{R}^D):=\left\{\mathbf{F}:=(F_1,\ldots,F_D):\ \text{for any } i\in\{1,\ldots,D\},\ F_i\in H^1(\mathbb{R}^D)\right\}$$

and, for any $\mathbf{F}\in H^1(\mathbb{R}^D;\mathbb{R}^D)$, let

$$\|\mathbf{F}\|_{H^1(\mathbb{R}^D;\mathbb{R}^D)}:=\left[\sum_{i=1}^n\|F_i\|^2_{H^1(\mathbb{R}^D)}\right]^{\frac12}.$$

The *vector-valued BMO space* $\mathrm{BMO}\,(\mathbb{R}^D;\mathbb{R}^D)$ is defined by setting

$$\begin{aligned}\mathrm{BMO}\,(\mathbb{R}^D;\mathbb{R}^D):=&\big\{\mathbf{G}:=(G_1,\ldots,G_D):\\&\qquad\text{for any } i\in\{1,\ldots,D\},\ G_i\in\mathrm{BMO}\,(\mathbb{R}^D)\big\}.\end{aligned}$$

**Theorem 2.4** ([6])**.** *Let* $\mathbf{F} \in H^1(\mathbb{R}^D, \mathbb{R}^D)$ *and* $\mathbf{G} \in \mathrm{BMO}\,(\mathbb{R}^D, \mathbb{R}^D)$ *be two vector fields such that* $\operatorname{curl} F = 0$ *and* $\operatorname{div} G = 0$*. Then the scalar product* $\mathbf{F} \cdot \mathbf{G} \in H^{\log}(\mathbb{R}^D)$ *in the distribution sense and there exists a positive constant* $C$*, independent of* $\mathbf{F}$ *and* $\mathbf{G}$*, such that*

$$\|\mathbf{F} \cdot \mathbf{G}\|_{H^{\log}(\mathbb{R}^D)} \le C \, \|\mathbf{F}\|_{H^1(\mathbb{R}^D;\, \mathbb{R}^D)} \, \|\mathbf{G}\|_{\mathrm{BMO}\,(\mathbb{R}^D;\mathbb{R}^D)} \, .$$

2.2. **Bilinear decompositions and commutators of singular integral operators on $\mathbb{R}^D$.** This subsection is devoted to the summarization of conclusions on bilinear decompositions and commutators of singular integral operators on $\mathbb{R}^D$.

We first recall the notions of Calderón-Zygmund operators from [31]; see also [46].

**Definition 2.5.** Let $\delta \in (0, 1]$. A continuous function

$$K : \left\{\mathbb{R}^D \times \mathbb{R}^D\right\} \setminus \{(x, x) : \ x \in \mathbb{R}^D\} \to \mathbb{C}$$

is called a *δ-Calderón-Zygmund kernel* if there exists a positive constant $C_{(K)}$, depending on $K$, such that, for all $x$, $y \in \mathbb{R}^D$ with $x \ne y$,

$$|K(x, y)| \le \frac{C_{(K)}}{|x - y|^D}$$

and, for all $x$, $\widetilde{x}$, $y \in \mathbb{R}^D$ with $2|x - \widetilde{x}| \le |x - y|$,

$$|K(x, y) - K\left(\widetilde{x}, y\right)| + |K(y, x) - K\left(y, \widetilde{x}\right)| \le C_{(K)} \frac{|x - \widetilde{x}|^\delta}{|x - y|^{D+\delta}}.$$

A linear operator $T : \ \mathscr{S}(\mathbb{R}^D) \to \mathscr{S}'(\mathbb{R}^D)$ is called a *δ-Calderón-Zygmund operator* if $T$ can be extended to a bounded linear operator on $L^2(\mathbb{R}^D)$ and if there exists a $\delta$-Calderón-Zygmund kernel $K$ such that, for all $f \in \mathscr{C}_c^\infty(\mathbb{R}^D)$ (the space of all smooth functions with compact supports) and all $x \notin \operatorname{supp}(f)$,

$$Tf(x) := \int_{\mathbb{R}^D} K(x, y) f(y)\, dy.$$

$T$ is called a *Calderón-Zygmund operator* if $T$ is a $\delta$-Calderón-Zygmund operator for some $\delta \in (0, 1]$.

In what follows, for any Calderón-Zygmund operator $T$ and its adjoint operator $T^*$, $T^*1 = 0$ and $T1 = 0$ represent, respectively, $\int_{\mathbb{R}^D} Ta(x)\, dx = 0$ and $\int_{\mathbb{R}^D} T^*a(x)\, dx = 0$ for all $(1, \infty)$-atoms $a$. Let $b$ be a locally integrable function on $\mathbb{R}^D$. Then $T^*b = 0$ means that $\int_{\mathbb{R}^D} b(x)Ta(x)\, dx = 0$ for all $(1, \infty)$-atoms $a$.

The theory of commutators of singular integrals, originated from the work of Coifman et al. [12], has been a vital part of the theory of singular integrals, which attracts a lot of attentions and has important appplications in harmonic analysis and partial differential equations; see, for example, [9, 10, 12, 24, 46]. Coifman et al. [12] showed that the commutator $[b, T]$ of a Calderón-Zygmund operator $T$ with a function $b \in \mathrm{BMO}\,(\mathbb{R}^D)$, defined by setting

$$[b, T](f)(x) := b(x)T(f)(x) - T(bf)(x), \quad \forall\, x \in \mathbb{R}^D,$$

is bounded on $L^p(\mathbb{R}^D)$ for all $p \in (1, \infty)$. From then on, there appeared a lot of literatures on the boundedness of commutators on various kinds of function spaces over different underlying spaces and their applications; see [17, 29, 31] for some classical results and fundamental tools in the theory of commutators.

We now recall a wavelet basis on $\mathbb{R}^D$ from [15]; see also [8]. Choose the father and the mother wavelets $\phi,\,\psi\in C^k(\mathbb{R})$ (the *set of all functions with continuous derivatives up to order* $k$) with compact supports such that $\widehat{\phi}(0)=(2\pi)^{-1/2}$ and, for each $l\in\{0,\,\dots,\,k\}$,

$$\int_{\mathbb{R}} x^l\psi(x)\,dx=0,$$

where $\widehat{\phi}$ denotes the *Fourier transform* of $\phi$, namely, for any $\xi\in\mathbb{R}^D$,

$$\widehat{\phi}(\xi):=(2\pi)^{-n/2}\int_{\mathbb{R}^D} e^{-ix\xi}\phi(x)\,dx.$$

The extension of the above wavelets from 1 dimension to $D$-dimension can be realized by the standard procedure of tensor products. Precisely, let

$$\vec{\theta}_D:=(\overbrace{0,\,\dots,\,0}^{D\text{ times}})\quad\text{and}\quad E:=\{0,\,1\}^D\setminus\{\vec{\theta}_D\}.$$

Let $\mathcal{D}_0$ be the set of all *dyadic cubes* in $\mathbb{R}^D$ with side lengths not bigger than 1, that is, for any $I\in\mathcal{D}_0$, there exist $j\in\mathbb{Z}_+$ and $k:=\{k_1,\,\dots,\,k_D\}\in\mathbb{Z}^n$ such that

$$I:=I_{j,\,k}:=\{x\in\mathbb{R}^D:\ k_i\le 2^jx_i<k_i+1\text{ for any }i\in\{1,\,\dots,\,D\}\}. \tag{2.3}$$

There exist two families $\{\phi_I\}_{I\in\mathcal{D}_0}$ and $\{\psi_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E}$ such that, for any $j\in\mathbb{Z}_+$, $k\in\mathbb{Z}^D$ (with $I=I_{j,\,k}\in\mathcal{D}_0$ as in (2.3)), $\lambda\in E\cup\{\vec{\theta}_D\}$ and $x\in\mathbb{R}^D$,

$$\Psi_I^\lambda(x):=\Psi_{I_{j,\,k}}^\lambda(x):=\begin{cases}\phi_{I_{j,\,k}}(x) & \text{when } j=0,\ k\in\mathbb{Z}^D\text{ and }\lambda=\vec{\theta}_D,\\ \psi^\lambda_{I_{j-1,\,k}}(x) & \text{when } j\in\mathbb{N},\ k\in\mathbb{Z}^D\text{ and }\lambda\in E,\\ 0 & \text{otherwise,}\end{cases}$$

$\{\Psi_I^\lambda\}_{I\in\mathcal{D}_0,\,\lambda\in E\cup\{\vec{\theta}_D\}}$ forms an orthonormal basis of $L^2(\mathbb{R}^D)$. Moreover, for any $j\in\mathbb{Z}_+$, let $V_j$ be the closed subspace of $L^2(\mathbb{R}^D)$ spanned by $\{\phi_I\}_{|I|=2^{-jn}}$. It is known that $\{V_j\}_{j\in\mathbb{Z}_+}$ is an MRA; see [8] for more details.

In what follows, let $L^{1,\infty}(\mathbb{R}^D)$ be the *weak* $L^1(\mathbb{R}^D)$ *space* defined by setting

$$L^{1,\infty}(\mathbb{R}^D):=\left\{f\text{ measurable}:\right.$$
$$\left.\|f\|_{L^{1,\infty}(\mathbb{R}^D)}:=\sup_{t\in(0,\infty)}\left[t\mu\left(\{x\in\mathbb{R}^D:\ |f(x)|>t\}\right)\right]<\infty\right\}.$$

Let $\mathcal{K}$ be the set of all sublinear operators $T$ bounded from $H^1(\mathbb{R}^D)$ into $L^1(\mathbb{R}^D)$ and from $L^1(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$ satisfying that there exists a positive constant $C$ such that, for all $g\in\mathrm{BMO}\,(\mathbb{R}^D)$ and $(1,\infty)$-atoms $a$,

$$\|(g-g_B)Ta\|_{L^1(\mathbb{R}^D)}\le C\|g\|_{\mathrm{BMO}\,(\mathbb{R}^D)},$$

where $g_B:=\frac{1}{|B|}\int_B g(x)\,dx$. The *sublinear commutator* $[b,T]$ is defined by setting

$$[b,T](f)(x):=T\left([b(x)-b(\cdot)]\,f(\cdot)\right)(x),\quad\forall\,x\in\mathbb{R}^D.$$

In what follows, the *bilinear operator* $\mathfrak{S}$ is defined by setting

$$\mathfrak{S}(f,g) := -\sum_{I\in\mathcal{D}_0}\sum_{\lambda\in E}\left\langle f,\,\Psi_I^\lambda\right\rangle\left\langle g,\,\Psi_I^\lambda\right\rangle\left(\Psi_I^\lambda\right)^2.$$

The following subbilinear decomposition of $[g,T](f)$ was claimed in [46, Theorem 3.1] without assuming that $T$ is bounded from $L^1(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$, whose proof has a gap; see [63, Theorem 11.2.7] for the following corrected version.

**Theorem 2.6** ([46, 63]). *Let $T\in\mathcal{K}$ be linear. Then there exists a bounded subbilinear operator $\mathcal{R}:=\mathcal{R}_T:\ H^1_{\rm at}(\mathcal{X})\times\mathrm{BMO}\,(\mathbb{R}^D)\to L^1(\mathbb{R}^D)$ such that, for all $f\in H^1(\mathbb{R}^D)$ and $g\in\mathrm{BMO}\,(\mathbb{R}^D)$,*

$$|T\left(\mathfrak{S}\left(f,g\right)\right)|-\mathcal{R}\left(f,g\right)\le|\left[g,T\right]\left(f\right)|\le|T\left(\mathfrak{S}\left(f,g\right)\right)|+\mathcal{R}\left(f,g\right)$$

*almost everywhere on $\mathbb{R}^D$.*

As an application of Theorem 2.6, Ky [46, Corollary 3.1] showed that $[g,T]$ is bounded from $H^1(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$.

**Corollary 2.7** ([46]). *Let $T\in\mathcal{K}$. Then the subbilinear operator, defined by setting*

$$\mathfrak{B}\left(f,g\right):=\left[g,T\right]\left(f\right)$$

*with $(f,g)\in H^1(\mathbb{R}^D)\times\mathrm{BMO}\,(\mathbb{R}^D)$, is bounded from $H^1(\mathbb{R}^D)\times\mathrm{BMO}\,(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$ and there exists a positive constant such that, for all $(f,g)\in H^1(\mathbb{R}^D)\times\mathrm{BMO}\,(\mathbb{R}^D)$,*

$$\|\mathfrak{B}\left(f,g\right)\|_{L^{1,\infty}(\mathbb{R}^D)}\le C\|f\|_{H^1(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}\,(\mathbb{R}^D)}.$$

*Particularly, the commutator $[g,T]$ is bounded from $H^1(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$.*

When $T\in\mathcal{K}$ is linear, the bilinear decomposition of $[g,T](f)$ was obtained in [46, Theorem 3.1].

**Theorem 2.8** ([46]). *Let $T\in\mathcal{K}$ be linear. Then there exists a bounded bilinear operator $\widetilde{\mathcal{R}}:=\widetilde{\mathcal{R}}_T:\ H^1(\mathbb{R}^D)\times\mathrm{BMO}\,(\mathbb{R}^D)\to L^1(\mathbb{R}^D)$ such that, for all $(f,g)\in H^1(\mathbb{R}^D)\times\mathrm{BMO}\,(\mathbb{R}^D)$,*

$$\left[g,T\right]\left(f\right)=\widetilde{\mathcal{R}}\left(f,g\right)+T\left(\mathfrak{S}\left(f,g\right)\right)$$

*holds true almost everywhere on $\mathbb{R}^D$.*

Then we display the result from [46, Theorem 3.3] (see also [63, Theorem 11.3.4]) that $[g,T]$ is bounded from the Hardy-type space $H^1_g(\mathbb{R}^D)$ into $L^1(\mathbb{R}^D)$, where $H^1_g(\mathbb{R}^D)$ was original defined in [46, p. 2933] which is recalled as follows.

**Definition 2.9.** Let $g$ be a non-constant $\mathrm{BMO}\,(\mathbb{R}^D)$-function. A function $f$ in $H^1(\mathbb{R}^D)$ is said to belong to the *Hardy-type space* $H^1_g(\mathbb{R}^D)$ if $[g,\mathcal{M}](f)$, defined by setting

$$[g,\mathcal{M}](f)(x):=\mathcal{M}\left(g(x)f(\cdot)-g(\cdot)f(\cdot)\right)(x),\quad\forall\,x\in\mathbb{R}^D,$$

belongs to $L^1(\mathbb{R}^D)$, where $\mathcal{M}$ is as in (2.2). Moreover, the *norm* of $f$ in $H^1_g(\mathbb{R}^D)$ is defined by setting

$$\|f\|_{H^1_g(\mathbb{R}^D)}:=\|f\|_{H^1(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}\,(\mathbb{R}^D)}+\|[g,\mathcal{M}](f)\|_{L^1(\mathbb{R}^D)}.$$

**Corollary 2.10** ([46,63]). *Let $g$ be a non-constant* $\mathrm{BMO}(\mathbb{R}^D)$*-function and $T \in \mathcal{K}$. Then the commutator $[g,T]$ is bounded from $H^1_g(\mathbb{R}^D)$ into $L^1(\mathbb{R}^D)$ and there exists a positive constant $C$ such that, for all $f \in H^1_g(\mathbb{R}^D)$,*

$$\|[g,T](f)\|_{L^1(\mathbb{R}^D)} \le C\|f\|_{H^1_g(\mathbb{R}^D)}.$$

**Remark 2.11.** By [46, Remark 5.1], we know that, for every Calderón-Zygmund operator $T$ and $g \in \mathrm{BMO}(\mathbb{R}^D)$, $[g,T]$ is bounded from $H^1_g(\mathbb{R}^D)$ into $L^1(\mathcal{X})$. Moreover, $H^1_g(\mathbb{R}^D)$ is the *biggest* space having this property.

Ky [46, Theorem 3.4] (see also [63, Theorem 11.4.10]) also gave the following strongly bilinear estimates which improve Corollary 2.7.

**Theorem 2.12** ([46,63]). *Let $T$ be a linear operator in $\mathcal{K}$. Assume that $I \in \mathbb{N}$, $A_i$ and $B_i$* with *$i \in \{1,\dots,I\}$ are Calderón-Zygmund operators which are bounded on $L^2(\mathbb{R}^D)$ and $A_i 1 = A_i^* 1 = 0 = B_i 1 = B_i^* 1$. Suppose that, for all $f$, $g \in L^2(\mathbb{R}^D)$,*

$$\int_{\mathcal{X}} \left[\sum_{i=1}^{I} A_i f(x) \cdot B_i g(x)\right] d\mu(x) = 0.$$

*Then the bilinear operator $\mathcal{I}$, defined by setting $\mathcal{I}(f,g) := \sum_{i=1}^{I}[B_i g, T](A_i f)$, is bounded from $H^1(\mathbb{R}^D) \times \mathrm{BMO}(\mathbb{R}^D)$ into $L^1(\mathbb{R}^D)$ and there exists a positive constant $C$ such that, for all $(f,g) \in H^1(\mathbb{R}^D) \times \mathrm{BMO}(\mathbb{R}^D)$,*

$$\|\mathcal{I}(f,g)\|_{L^{1,\infty}(\mathbb{R}^D)} \le C\|f\|_{H^1(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}(\mathbb{R}^D)}.$$

It was shown in [46, Theorem 3.5] (see also [63, Theorem 11.4.11]) that the linear commutator $[b,T]$ is bounded from $H^1_g(\mathbb{R}^D)$ into $h^1(\mathbb{R}^D)$ (see [30] or (3.2) below for the definition of $h^1(\mathbb{R}^D)$).

**Theorem 2.13** ([46, 63]). *Let $g$ be a non-constant* $\mathrm{BMO}(\mathbb{R}^D)$*-function and let $T$ be a Calderón-Zygmund operator, which is bounded on $L^2(\mathbb{R}^D)$, satisfying $T^*1 = T^*g = 0$. Then the commutator $[b,T]$ maps continuously from $H^1_g(\mathbb{R}^D)$ into $h^1(\mathbb{R}^D)$ and, moreover, there exists a positive constant $C$ such that, for all $f \in H^1_g(\mathbb{R}^D)$,*

$$\|[g,T](f)\|_{h^1(\mathbb{R}^D)} \le C\|f\|_{H^1_g(\mathbb{R}^D)}.$$

Moreover, [46, Theorem 3.6] (see also [63, Theorem 11.4.12]) provides a sufficient condition that the linear commutator $[b,T]$ maps continuously from $H^1_g(\mathbb{R}^D)$ into $H^1(\mathbb{R}^D)$.

**Theorem 2.14** ([46, 63]). *Let $g$ be a non-constant* $\mathrm{BMO}^{\log}(\mathbb{R}^D)$*-function and let $T$ be a Calderón-Zygmund operator, which is bounded on $L^2(\mathbb{R}^D)$, satisfying $T^*1 = T^*g = 0$. Then the commutator $[b,T]$ maps continuously from $H^1_g(\mathbb{R}^D)$ into $H^1(\mathbb{R}^D)$ and, moreover, there exists a positive constant $C$ such that, for all $f \in H^1_g(\mathbb{R}^D)$,*

$$\|[g,T](f)\|_{H^1(\mathbb{R}^D)} \le C\|f\|_{H^1_g(\mathbb{R}^D)}.$$

2.3. **Bilinear decompositions for products of functions in $H^1_{\mathcal{L}}(\mathbb{R}^D)$ and $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$.** In this subsection, we display the results in Ky [48] of the bilinear decompositions for products of functions in $H^1_{\mathcal{L}}(\mathbb{R}^D)$ and $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$, where $\mathcal{L} := -\triangle + V$ is the *Schrödinger operator* associated to a non-negative potential $V \neq 0$. A non-negative locallly integrable potential $V$ is said to belong to the *reverse Hölder class* $\mathrm{RH}_q(\mathbb{R}^D)$, $q \in (1,\infty)$, if there exists a positive constant $C$ such that, for all balls $B$ of $\mathbb{R}^D$,

$$\left\{\frac{1}{|B|}\int_B [V(x)]^q\,dx\right\}^{1/q} \le \frac{C}{|B|}\int_B V(x)\,dx.$$

Let us now recall the notions of the *Hardy space* $H^1_{\mathcal{L}}(\mathbb{R}^D)$ and the BMO *space* $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$ associated to the Schrödinger operator $\mathcal{L}$. Let $\{T_t\}_{t\in(0,\infty)}$ be a semigroup generated by $\mathcal{L}$ and $\{T_t(\cdot,\cdot)\}_{t\in(0,\infty)}$ their kernels, that is, for all $t \in (0,\infty)$, $f \in L^2(\mathbb{R}^D)$ and $x \in \mathbb{R}^D$,

$$T_t f(x) := e^{-t\mathcal{L}} f(x) := \int_{\mathbb{R}^D} T_t(x,y) f(y)\,dy.$$

A function $f \in L^2(\mathbb{R}^D)$ is said to belong to the *space* $\mathbb{H}^1_{\mathcal{L}}(\mathbb{R}^D)$ if

$$\|f\|_{\mathbb{H}^1_{\mathcal{L}}(\mathbb{R}^D)} := \|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^D)} < \infty,$$

where $\mathcal{M}_{\mathcal{L}} f(x) := \sup_{t\in(0,\infty)} |T_t f(x)|$ for all $x \in \mathbb{R}^D$. The *Hardy-type space* $H^1_{\mathcal{L}}(\mathbb{R}^D)$ is defined as the completion of $\mathbb{H}^1_{\mathcal{L}}(\mathbb{R}^D)$ with respect to the above norm.

In what follows, for any $x \in \mathbb{R}^D$ and $r \in (0,\infty)$, let

$$B(x,r) := \left\{y \in \mathbb{R}^D :\ |x-y| < r\right\}.$$

It was shown in [21, Theorem 4] that the dual space of $H^1_{\mathcal{L}}(\mathbb{R}^D)$ is the BMO-*type space* $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$ which consists of all functions in $\mathrm{BMO}\,(\mathbb{R}^D)$ such that

$$\|f\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)} := \|f\|_{\mathrm{BMO}\,(\mathbb{R}^D)} + \sup_{x\in\mathbb{R}^D,\, r\in[\rho(x),\infty)} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy < \infty,$$

where $\rho$ is an *admissible function* defined by setting

$$\rho(x) := \sup\left\{r \in (0,\infty) :\ \frac{1}{r^{d-2}}\int_{B(x,r)} V(y)\,dy \le 1\right\}.$$

The following [48, Theorem 1.1] essentially improves the conclusion of [52, Theorem 1].

**Theorem 2.15** ([48])**.** *There exist two bounded bilinear operators:*

$$\mathscr{L}_{\mathcal{L}} :\ H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D) \to L^1(\mathbb{R}^D),$$

$$\mathscr{H}_{\mathcal{L}} :\ H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D) \to H^{\log}(\mathbb{R}^D),$$

*and a positive constant $C$ such that, for all $f \in H^1(\mathbb{R}^D)$ and $g \in \mathrm{BMO}\,(\mathbb{R}^D)$,*

$$f \times g = \mathscr{L}_{\mathcal{L}}(f,g) + \mathscr{H}_{\mathcal{L}}(f,g)$$

*and*

$$\|\mathscr{L}_{\mathcal{L}}(f,g)\|_{L^1(\mathbb{R}^D)} + \|\mathscr{H}_{\mathcal{L}}(f,g)\|_{H^{\log}(\mathbb{R}^D)} \le C\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)}.$$

Analogous to [7, Theorem 1.8], Ky [48] showed that the following result, associated with the Schrödinger operator, also holds true.

**Theorem 2.16** ([48]). *Let $f \in H^1_{\mathcal{L}}(\mathbb{R}^D)$ and $g \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^D)$. Then, for almost every $x \in \mathbb{R}^D$,*

$$f(x)g(x) = \lim_{\epsilon\to 0}(f \times g)_\epsilon(x).$$

2.4. **Endpoint estimates for commutators of singular integrals associated to Schrödinger operators on $\mathbb{R}^D$.** This subsection aims to summarize the endpoint boundedness of commutators associated to the Schrödinger operator $\mathcal{L}$ via bilinear decompositions discussed in Subsection 2.3.

Let $q \in (1,\infty]$ and $\epsilon \in (0,\infty)$. Recall that a function $a$ is called a *generalized* $(H^1_{\mathcal{L}}(\mathbb{R}^D), q, \epsilon)$*-atoms* related to the ball $B(x_0,r)$ with $(x_0,r) \in \mathbb{R}^D \times (0,\infty)$ if

i) $\mathrm{supp}\,(a) \subset B(x_0,r)$;
ii) $\|a\|_{L^q(\mathbb{R}^D)} \le |B(x_0,r)|^{1/q-1}$;
iii) $|\int_{\mathbb{R}^D} a(x)\,d\mu(x)| \le [\frac{r}{\rho(x_0)}]^\epsilon$.

Let $\mathcal{K}_{\mathcal{L}}$ be the set of all sublinear operators T which map continuously from $H^1_{\mathcal{L}}(\mathbb{R}^D)$ into $L^1(\mathbb{R}^D)$ and satisfy that there exist $q \in (1,\infty]$ and $\epsilon \in (0,\infty)$ such that, for any $g \in \mathrm{BMO}\,(\mathbb{R}^D)$ and generalized $(H^1_{\mathcal{L}}(\mathbb{R}^D), q, \epsilon)$-atom $a$ related to a ball $B$, $\|(g - g_B)Ta\|_{L^1(\mathcal{X})} \le C$, where $C$ is a positive constant independent of $g$ and $a$.

Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^D)$ and $T \in \mathcal{K}_{\mathcal{L}}$. The *sublinear commutator* $[g,T]$ is defined by setting

$$[g,T](f)(x) := T\left([b(x) - b(\cdot)]\, f(\cdot)\right)(x), \quad \forall\, x \in \mathbb{R}^D.$$

The following subbilinear decomposition of $[g,T](f)$ was obtained in [50, Theorem 3.1].

**Theorem 2.17** ([50]). *Let $T \in \mathcal{K}_{\mathcal{L}}$ be bounded from $L^1(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$. Then there exists a bounded subbilinear operator $\mathcal{R} := \mathcal{R}_T :\ H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D) \to L^1(\mathbb{R}^D)$ such that, for all $f \in H^1_{\mathcal{L}}(\mathbb{R}^D)$ and $g \in \mathrm{BMO}\,(\mathbb{R}^D)$,*

$$|T\left(\mathfrak{S}\,(f,g)\right)| - \mathcal{R}\,(f,g) \le |[g,T]\,(f)| \le |T\left(\mathfrak{S}\,(f,g)\right)| + \mathcal{R}\,(f,g)$$

*almost everywhere on $\mathbb{R}^D$, where $\mathfrak{S}:\ H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D) \to L^1(\mathbb{R}^D)$ is a bilinear operator.*

Applying Theorem 2.17, Ky [50, Proposition 3.1] showed that $[g,T](f)$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^D)$ to $L^{1,\infty}(\mathbb{R}^D)$.

**Corollary 2.18** ([50]). *Let $T \in \mathcal{K}_{\mathcal{L}}$ be bounded from $L^1(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$. Then the subbilinear operator $\mathfrak{B}(f,g)$, defined by setting*

$$\mathfrak{B}\,(f,g) := [g,T]\,(f)$$

*with $(f,g) \in H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D)$, is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D)$ into $L^{1,\infty}(\mathbb{R}^D)$ and, moreover, there exists a positive constant $C$ such that, for all $(f,g) \in$*

$H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D)$,

$$\|\mathfrak{B}\,(f,g)\|_{L^{1,\infty}(\mathbb{R}^D)} \le C\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}\,(\mathbb{R}^D)}.$$

*Particularly, the commutator* $[g,T]$ *is bounded from* $H^1_{\mathcal{L}}(\mathbb{R}^D)$ *to* $L^{1,\infty}(\mathbb{R}^D)$.

If $T \in \mathcal{K}_{\mathcal{L}}$ is linear, then the bilinear decomposition of $[g,T]$ was established in [50, Theorem 3.2].

**Theorem 2.19** ([50])**.** *Let* $T \in \mathcal{K}_{\mathcal{L}}$ *be a bounded linear operator from* $L^1(\mathbb{R}^D)$ *into* $L^{1,\infty}(\mathbb{R}^D)$. *Then there exists a bounded bilinear operator*

$$\widetilde{\mathcal{R}} := \widetilde{\mathcal{R}}_T : \ H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D) \to L^1(\mathbb{R}^D)$$

*such that, for all* $(f,g) \in H^1_{\mathcal{L}}(\mathbb{R}^D) \times \mathrm{BMO}\,(\mathbb{R}^D)$,

$$[g,T]\,(f) = \widetilde{\mathcal{R}}\,(f,g) + T\,(\mathfrak{S}\,(f,g))$$

*holds true almost everywhere on* $\mathbb{R}^D$.

We now recall the notions of $\mathcal{L}$-Calderón-Zygmund operators from [50]. Let $\delta \in (0,1]$. A continuous function $K:\ \{\mathbb{R}^D \times \mathbb{R}^D\} \setminus \{(x,x):\ x \in \mathbb{R}^D\} \to \mathbb{C}$ is called a $(\delta,\mathcal{L})$-*Calderón-Zygmund kernel* if, for each $N \in (0,\infty)$, there exists a positive constant $C_{(N)}$, depending on $N$, such that

$$|K(x,y)| \le \frac{C_{(N)}}{|x-y|^D}\left[1 + \frac{|x-y|}{\rho(x)}\right]^{-N}$$

for all $x$, $y \in \mathbb{R}^D$ with $x \neq y$, and

$$|K(x,y) - K\,(\widetilde{x},y)| + |K(y,x) - K\,(y,\widetilde{x})| \le C_{(N)}\frac{|x-\widetilde{x}|^\delta}{|x-y|^{D+\delta}}$$

for all $x$, $\widetilde{x}$, $y \in \mathbb{R}^D$ with $2|x-\widetilde{x}| \le |x-y|$.

A linear operator $T:\ \mathscr{S}(\mathbb{R}^D) \to \mathscr{S}'(\mathbb{R}^D)$ is called a $(\delta,\mathcal{L})$-*Calderón-Zygmund operator* if there exists a $(\delta,\mathcal{L})$-Calderón-Zygmund kernel $K$ such that, for all $f \in C^\infty_c(\mathbb{R}^D)$ and all $x \notin \mathrm{supp}\,(f)$,

$$Tf(x) := \int_{\mathbb{R}^D} K(x,y) f(y)\,dy.$$

A linear operator $T$ is called a $\mathcal{L}$-*Calderón-Zygmund operator* if it is a $(\delta,\mathcal{L})$-Calderón-Zygmund operator for some $\delta \in (0,1]$. We say that $T$ satisfies $T^*1 = 0$ if there exist $q \in (1,\infty]$ and $\epsilon \in (0,\infty)$ such that $\int_{\mathbb{R}^D} Ta(x) = 0$ for every generalized $(H^1_{\mathcal{L}}(\mathbb{R}^D),q,\epsilon)$-atom $a$.

For any $\theta \in [0,\infty)$, denote by $\mathrm{BMO}^{\log}_{\mathcal{L},\theta}(\mathbb{R}^D)$ the *set of all locally integral functions* $f$ such that

$$\|f\|_{\mathrm{BMO}^{\log}_{\mathcal{L},\theta}(\mathbb{R}^D)} := \sup_{x\in\mathbb{R}^D,\, r\in(0,\infty)} \left\{\frac{\log(e+\frac{\rho(x)}{r})}{[1+\frac{r}{\rho(x)}]^\theta}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_B|\,dy\right\} < \infty.$$

Write $\mathrm{BMO}^{\log}_{\mathcal{L},0}(\mathbb{R}^D)$ simply by $\mathrm{BMO}^{\log}_{\mathcal{L}}(\mathbb{R}^D)$.

As applications of bilinear decompositions of $[g,T]$ in Theorem 2.19, the boundedness on the Hardy space $H^{\log}(\mathbb{R}^D)$ of linear commutators were obtained in [50, Theorem 3.3].

**Theorem 2.20** ([50]). (i) *$T$ be an $\mathcal{L}$-Calderón-Zygmund operator satisfying $T^*1=0$ and let $g\in \mathrm{BMO}^{\log}_{\mathcal{L}}(\mathbb{R}^D)$. Then the linear commutator $[g,T]$ is bounded on $H^1_{\mathcal{L}}(\mathbb{R}^D)$ and, moreover, there exists a positive constant $C$ such that, for all $f\in H^1_{\mathcal{L}}(\mathbb{R}^D)$,*

$$\|[g,T](f)\|_{H^1_{\mathcal{L}}(\mathbb{R}^D)}\le C\|g\|_{\mathrm{BMO}^{\log}_{\mathcal{L}}(\mathbb{R}^D)}\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^D)}.$$

(ii) *If $V\in \mathrm{RH}_d(\mathbb{R}^D)$, then the converse of* (i) *also holds true. That is, if $b\in \mathrm{BMO}\,(\mathbb{R}^D)$ and $[b,T]$ is bounded on $H^1_{\mathcal{L}}(\mathbb{R}^D)$ for every $\mathcal{L}$-Calderón-Zygmund operator satisfying $T^*1=0$, then $b\in\mathrm{BMO}^{\log}_{\mathcal{L}}(\mathbb{R}^D)$. Moreover,*

$$\|b\|_{\mathrm{BMO}^{\log}_{\mathcal{L}}(\mathbb{R}^D)}\sim \|b\|_{\mathrm{BMO}\,(\mathbb{R}^D)}+\sum_{i=1}^{D}\|[b,R_j]\|_{H^1_{\mathcal{L}}(\mathbb{R}^D)\to H^1_{\mathcal{L}}(\mathbb{R}^D)},$$

*where the equivalent positive constants are independent of $b$, $\{R_i\}_{i=1}^D$ are Riesz transforms and $\|\cdot\|_{H^1_{\mathcal{L}}(\mathbb{R}^D)\to H^1_{\mathcal{L}}(\mathbb{R}^D)}$ denotes the* operator norm *on $H^1_{\mathcal{L}}(\mathbb{R}^D)$.*

## 3. Bilinear decompositions for products of local Hardy and Lipschitz or bmo spaces on $\mathbb{R}^D$

In this section, we discuss the bilinear decompositions for products of functions in local Hardy spaces $h^p(\mathbb{R}^D)$ and local Lipschitz spaces $\Lambda_\alpha(\mathbb{R}^D)$ and their applications to the div-curl lemmas.

For any $m\in\mathbb{N}$, $f\in\mathscr{S}'(\mathbb{R}^D)$ and $x\in\mathbb{R}^D$, let

$$f^*_{m,\,\mathrm{loc}}(x):=\sup_{\varphi\in\mathscr{S}_m(\mathbb{R}^D)}\ \sup_{\substack{|y-x|<t\\ t\in(0,\,1)}}|f*\varphi_t(y)|\,, \tag{3.1}$$

where $\mathscr{S}_m(\mathbb{R}^D)$ is as in (2.1).

Then, for any $p\in(0,1]$, the *local Hardy space* $h^p(\mathbb{R}^D)$ is defined by setting

$$h^p(\mathbb{R}^D):=\left\{f\in\mathcal{S}'(\mathbb{R}^D):\ \|f\|_{h^p(\mathbb{R}^D)}:=\|f^*_{m,\,\mathrm{loc}}\|_{L^p(\mathbb{R}^D)}<\infty\right\}, \tag{3.2}$$

see [30] for more properties of $h^p(\mathbb{R}^D)$.

Recall that, in [4], Bonami and Feuto introduced the following *variant local Orlicz-Hardy space* $h^\Phi_*(\mathbb{R}^D)$, defined by setting

$$h^\Phi_*(\mathbb{R}^D):=\left\{f\in\mathscr{S}'(\mathbb{R}^D):\ \|f\|_{h^\Phi_*(\mathbb{R}^D)}:=\|f^*_{\mathrm{loc}}\|_{L^\Phi_*(\mathbb{R}^D)}<\infty\right\}, \tag{3.3}$$

where $f^*_{\mathrm{loc}}$ is defined as in (3.1) with some $m\in\mathbb{N}\cap(\lfloor D(1/p-1)\rfloor,\infty)$, $\Phi$ as in (1.1) and, for any measurable function $g$,

$$\|g\|_{L^\Phi_*(\mathbb{R}^D)}:=\sum_{j\in\mathbb{Z}^n}\|g\|_{L^\Phi(\mathbb{Q}_j)}$$

with $j := (j_1, \ldots, j_D)$, $\mathbb{Q}_j := [j_1, j_1 + 1) \times \cdots \times [j_D, j_D + 1)$ and

$$\|g\|_{L^\Phi(\mathbb{Q}_j)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{Q}_j} \Phi\left(\frac{|g(x)|}{\lambda}\right) dx \le 1\right\}.$$

Obviously, $L^1(\mathbb{R}^D) \subset L_*^\Phi(\mathbb{R}^D)$ implies that $h^1(\mathbb{R}^D) \subset h_*^\Phi(\mathbb{R}^D)$.

Cao et al. [8] studied the bilinear decompositions of products of functions in local Hardy spaces $h^p(\mathbb{R}^D)$ and their dual spaces in the case when $p < 1$ and near to 1. Let $p \in (\frac{D}{D+1}, 1]$ and $\alpha := D(\frac{1}{p} - 1)$. The main result in [8] is the following two bilinear decompositions, which are extensions of corresponding linear decompositions (1.5) and (1.6) from [4], respectively.

**Theorem 3.1** ([8]). *Let $p \in (\frac{D}{D+1}, 1)$, $\alpha_0 := D(\frac{1}{p} - 1)$ and $\Phi$ be as in* (1.1). *Then*

(i) *there exist two bounded bilinear operators $S: h^p(\mathbb{R}^D) \times \Lambda_{\alpha_0}(\mathbb{R}^D) \to L^1(\mathbb{R}^D)$ and $T: h^p(\mathbb{R}^D) \times \Lambda_{\alpha_0}(\mathbb{R}^D) \to h^p(\mathbb{R}^D)$, and a positive constant $C$, such that, for any $(f, g) \in h^p(\mathbb{R}^D) \times \Lambda_{\alpha_0}(\mathbb{R}^D)$,*

$$f \times g = S(f, g) + T(f, g) \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^D)$$

*and*

$$\|S(f, g)\|_{L^1(\mathbb{R}^D)} + \|T(f, g)\|_{h^p(\mathbb{R}^D)} \le C\|f\|_{h^p(\mathbb{R}^D)}\|g\|_{\Lambda_{\alpha_0}(\mathbb{R}^D)};$$

(ii) *there exist two bounded bilinear operators $S: h^1(\mathbb{R}^D) \times \mathrm{BMO}(\mathbb{R}^D) \to L^1(\mathbb{R}^D)$ and $T: h^1(\mathbb{R}^D) \times \mathrm{BMO}(\mathbb{R}^D) \to h_*^\Phi(\mathbb{R}^D)$, and a positive constant $C$ such that, for any $(f, g) \in h^1(\mathbb{R}^D) \times \mathrm{BMO}(\mathbb{R}^D)$,*

$$f \times g = S(f, g) + T(f, g) \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^D)$$

*and*

$$\|S(f, g)\|_{L^1(\mathbb{R}^D)} + \|T(f, g)\|_{h_*^\Phi(\mathbb{R}^D)} \le C\|f\|_{h^1(\mathbb{R}^D)}\|g\|_{\mathrm{BMO}(\mathbb{R}^D)}.$$

As an application of Theorem 3.1, Cao et al. [8] obtained a div-curl lemma at the endpoint case $q = \infty$. Let

$$h^1(\mathbb{R}^D; \mathbb{R}^D) := \left\{\mathbf{F} := (F_1, \ldots, F_D) : \text{ for any } i \in \{1, \ldots, n\},\ F_i \in h^1(\mathbb{R}^D)\right\}$$

and, for any $\mathbf{F} \in h^1(\mathbb{R}^D; \mathbb{R}^D)$, let

$$\|\mathbf{F}\|_{h^1(\mathbb{R}^D; \mathbb{R}^D)} := \left[\sum_{i=1}^n \|F_i\|_{h^1(\mathbb{R}^D)}^2\right]^{\frac{1}{2}}.$$

The *local vector-valued BMO space* $\mathrm{bmo}(\mathbb{R}^D; \mathbb{R}^D)$ is defined by setting

$$\mathrm{bmo}(\mathbb{R}^D; \mathbb{R}^D) := \left\{\mathbf{G} := (G_1, \ldots, G_D) : \text{ for any } i \in \{1, \ldots, n\},\ G_i \in \mathrm{BMO}(\mathbb{R}^D)\right\}.$$

**Theorem 3.2** ([8]). *Let $\mathbf{F} \in h^1(\mathbb{R}^D; \mathbb{R}^D)$ with* $\operatorname{curl} \mathbf{F} \equiv 0$ *in the sense of distributions and*

$$\mathbf{G} \in \mathrm{bmo}(\mathbb{R}^D; \mathbb{R}^D)$$

*with* $\operatorname{div} \mathbf{G} \equiv 0$ *in the sense of distributions. Then $\mathbf{F} \cdot \mathbf{G} \in h_*^\Phi(\mathbb{R}^D)$, where $h_*^\Phi(\mathbb{R}^D)$ denotes the variant local Orlicz-Hardy space defined as in* (3.3) *above with $\Phi$ as in*

(1.1)*, and, moreover, there exists a positive constant* $C$*, independent of* $\mathbf{F}$ *and* $\mathbf{G}$*, such that*

$$\|\mathbf{F}\cdot\mathbf{G}\|_{H^{\log}(\mathbb{R}^D)} \le C\,\|\mathbf{F}\|_{H^1(\mathbb{R}^D;\,\mathbb{R}^D)}\,\|\mathbf{G}\|_{\mathrm{BMO}\,(\mathbb{R}^D;\mathbb{R}^D)}\,.$$

This result essentially improves the corresponding div-curl lemmas in [5,6].

## 4. Products of functions in $H^1$ and BMO on spaces of homogeneous type

The aims of Section 4 are twofold. The first aim is devoted to a survey of bilinear decompositions of products of functions in $H^1_{\mathrm{at}}(\mathcal{X})$ and BMO$\,(\mathcal{X})$ and their applications on a space $\mathcal{X}$ of homogeneous type and the second aim is to provide a new proof of Theorem 4.9 below.

Throughout this section, for the presentation simplicity, we *always assume* that $(\mathcal{X},d,\mu)$ is a metric measure space of homogeneous type, $\mathrm{diam}\,(\mathcal{X})=\infty$ and $(\mathcal{X},d,\mu)$ is non-atomic, namely, $\mu(\{x\})=0$ for any $x\in\mathcal{X}$. It is known that $\mu(\mathcal{X})=\infty$ if $\mathrm{diam}\,(\mathcal{X})=\infty$ (see, for instance, [2, Lemma 8.1]).

### 4.1. Bilinear decompositions for the products of functions in $H^1_{\mathrm{at}}(\mathcal{X})$ and BMO$\,(\mathcal{X})$.

In this subsection, we mainly review some known results from [27] on bilinear decompositions of products of functions in $H^1_{\mathrm{at}}(\mathcal{X})$ and BMO$\,(\mathcal{X})$ over a space $\mathcal{X}$ of homogeneous type.

To this end, we first recall the notion of the space of all test functions on $\mathcal{X}$, whose following versions were introduced by Han, Müller and Yang [35, Definition 2.2] (see also [36, Definition 2.8]).

**Definition 4.1.** Let $x_1\in\mathcal{X}$, $r\in(0,\infty)$, $\varrho\in(0,1]$ and $\vartheta\in(0,\infty)$. A function $f$ on $\mathcal{X}$ is said to belong to the *space of all test functions,* $\mathcal{G}(x_1,r,\varrho,\vartheta)$, if there exists a non-negative constant $\widetilde{C}$ such that

(T1) $|f(x)|\le \widetilde{C}\frac{1}{V_r(x_1)+V(x_1,x)}[\frac{r}{r+d(x_1,x)}]^{\gamma}$ for all $x\in\mathcal{X}$;

(T2) $|f(x)-f(y)|\le \widetilde{C}[\frac{d(x,y)}{r+d(x_1,x)}]^{\varrho}\frac{1}{V_r(x_1)+V(x_1,x)}[\frac{r}{r+d(x_1,x)}]^{\vartheta}$ for all $x,\ y\in\mathcal{X}$ satisfying $d(x,y)\le[r+d(x_1,x)]/2$,

here and hereafter,

$$V_r(x_1):=\mu(B(x_1,r))\quad\text{and}\quad V(x_1,x):=\mu(B(x_1,d(x_1,x))).$$

Moreover, for any $f\in\mathcal{G}(x_1,r,\varrho,\vartheta)$, its *nor*m is defined by setting

$$\|f\|_{\mathcal{G}(x_1,\,r,\,\varrho,\,\vartheta)}:=\inf\left\{\widetilde{C}:\ \widetilde{C}\text{ satisfies (T1) and (T2)}\right\}.$$

Fix $x_1\in\mathcal{X}$. It is easy to see that $\mathcal{G}(x_1,1,\varrho,\vartheta)$ is a Banach space. In what follows, we write $\mathcal{G}(x_1,1,\varrho,\vartheta)$ simply by $\mathcal{G}(\varrho,\vartheta)$.

For any given $\epsilon\in(0,1]$ and $\varrho,\vartheta\in(0,\epsilon]$, let $\mathcal{G}^{\epsilon}_0(\varrho,\vartheta)$ be the completion of the set $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\varrho,\vartheta)$. Moreover, for any $f\in\mathcal{G}^{\epsilon}_0(\varrho,\vartheta)$, let $\|f\|_{\mathcal{G}^{\epsilon}_0(\varrho,\vartheta)}:=\|f\|_{\mathcal{G}(\varrho,\vartheta)}$. Recall that the *dual space* $(\mathcal{G}^{\epsilon}_0(\varrho,\vartheta))'$ is defined to be the set of all continuous linear functionals $\mathcal{L}$ from $\mathcal{G}^{\epsilon}_0(\varrho,\vartheta)$ to $\mathbb{C}$, equipped with the weak-$*$ topology.

We point out that, for any $x\in\mathcal{X}$ and $r\in(0,\infty)$, $\mathcal{G}(x,r,\varrho,\vartheta)=\mathcal{G}(x_1,1,\varrho,\vartheta)$ with equivalent norms and the equivalent positive constants depending on $x$ and $r$.

The following notion of the space BMO$\,(\mathcal{X})$ is from [14].

**Definition 4.2.** (i) The *space* BMO$\,(\mathcal{X})$ is defined to be the class of all functions $g \in L^1_{\mathrm{loc}}(\mathcal{X})$ satisfying

$$\|g\|_{\mathrm{BMO}\,(\mathcal{X})} := \sup_B \frac{1}{\mu(B)} \int_B |g(x) - m_B(g)|\, d\mu(x) < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$ and

$$m_B(g) := \frac{1}{\mu(B)} \int_B g(x)\, d\mu(x).$$

(ii) Let $q \in (0, \infty]$. A function $g \in L^q_{\mathrm{loc}}(\mathcal{X})$ is said to belong to the *space* BMO$^q(\mathcal{X})$ if

$$\|g\|_{\mathrm{BMO}^q(\mathcal{X})} := \sup_B \left\{ \frac{1}{\mu(B)} \int_B |g(x) - m_B(g)|^q\, d\mu(x) \right\}^{1/q} < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$.

**Remark 4.3.** It was shown in [14] that the space BMO$^q(\mathcal{X})$ with $q \in (1, \infty)$ and BMO$\,(\mathcal{X})$ coincide with equivalent norms.

Now we recall the following notion of Hardy spaces $H^1_{\mathrm{at}}(\mathcal{X})$, which was originally introduced in [14].

**Definition 4.4.** Let $q \in (1, \infty]$. A function $a$ on $\mathcal{X}$ is called a $(1, q)$-*atom* if

(i) $\mathrm{supp}\,(a) \subset B$ for some ball $B \subset \mathcal{X}$;
(ii) $\|a\|_{L^q(\mathcal{X})} \le [\mu(B)]^{1/q-1}$;
(iii) $\int_{\mathcal{X}} a(x)\, d\mu(x) = 0$.

A function $f \in L^1(\mathcal{X})$ is said to be in the *Hardy space* $H^{1,\,q}_{\mathrm{at}}(\mathcal{X})$ if there exist $(1, q)$-atoms $\{a_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that

$$f = \sum_{j=1}^\infty \lambda_j a_j, \tag{4.1}$$

which converges in $L^1(\mathcal{X})$, and

$$\sum_{j=1}^\infty |\lambda_j| < \infty.$$

Moreover, the *norm* of $f$ in $H^{1,\,q}_{\mathrm{at}}(\mathcal{X})$ is defined by setting

$$\|f\|_{H^{1,\,q}_{\mathrm{at}}(\mathcal{X})} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| \right\},$$

where the infimum is taken over all possible decompositions of $f$ as in (4.1).

It was proved in [14] that, for any $q \in (1, \infty)$, $H^{1,\,q}_{\mathrm{at}}(\mathcal{X})$ coincides with $H^{1,\,\infty}_{\mathrm{at}}(\mathcal{X})$ in the sense of equivalent norms. Thus, from now on, $H^{1,\,q}_{\mathrm{at}}(\mathcal{X})$ is simply denoted by $H^1_{\mathrm{at}}(\mathcal{X})$.

**Remark 4.5.** Coifman and Weiss [14] showed that $H^1_{\mathrm{at}}(\mathcal{X})$ is a Banach space and its dual space is BMO$\,(\mathcal{X})$.

We also need to recall some notions and results from [49]. For a fixed $x_1 \in \mathcal{X}$, let

$$\theta_0(x,t) := \frac{t}{\log(e+t) + \log(e + d(x,x_1))}.$$

Let $L^{\log}(\mathcal{X})$ denote the *Musielak-Orlicz-type space* of all $\mu$-measurable functions $f$ such that

$$\int_{\mathcal{X}} \theta_0(x, |f(x)|)\, d\mu(x) < \infty;$$

see [49]. For any $f \in L^{\log}(\mathcal{X})$, the norm of $f$ in $L^{\log}(\mathcal{X})$ is defined by setting

$$\|f\|_{L^{\log}(\mathcal{X})} := \inf \left\{ \lambda \in (0,\infty) : \int_{\mathcal{X}} \theta_0 \left( x, \frac{|f(x)|}{\lambda} \right) d\mu(x) \le 1 \right\}.$$

**Remark 4.6.** It is clear that $L^1(\mathcal{X}) \subset L^{\log}(\mathcal{X})$ and, for all $f \in L^1(\mathcal{X})$,

$$\|f\|_{L^{\log}(\mathcal{X})} \le \|f\|_{L^1(\mathcal{X})}.$$

Let $\epsilon \in (0,1]$, $\varrho$, $\vartheta \in (0,\epsilon]$ and $f \in (\mathcal{G}_0^\epsilon(\varrho,\vartheta))'$. The *grand maximal function* $\mathcal{M}(f)$ is defined by setting, for all $x \in \mathcal{X}$,

$$\begin{aligned}(4.2)\qquad \mathcal{M}(f)(x) := \sup \big\{ |\langle f, h \rangle| :\ & h \in \mathcal{G}_0^\epsilon(\varrho,\vartheta), \\ & \|h\|_{\mathcal{G}(x,\,r,\,\varrho,\,\vartheta)} \le 1 \text{ for some } r \in (0,\infty) \big\}.\end{aligned}$$

The following notion of Musielak-Orlicz-type Hardy spaces is from [49].

**Definition 4.7.** Let $\epsilon \in (0,1]$ and $\varrho$, $\vartheta \in (0,\epsilon]$. The *Hardy space* $H^{\log}(\mathcal{X})$ is defined by setting

$$H^{\log}(\mathcal{X}) := \left\{ f \in (\mathcal{G}_0^\epsilon(\varrho,\vartheta))' :\ \|f\|_{H^{\log}(\mathcal{X})} := \|\mathcal{M}(f)\|_{L^{\log}(\mathcal{X})} < \infty \right\}.$$

We now recall the result in [49, Proposition 3.1].

**Lemma 4.8** ([49]). *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type, $\varrho \in (0,1]$ and $\vartheta \in (0,\infty)$. Then, for all $h \in \mathcal{G}(\varrho,\vartheta)$, there exists a positive constant $C$, independent of $h$, such that, for any $g \in \mathrm{BMO}\,(\mathcal{X})$,*

$$\|hg\|_{\mathrm{BMO}\,(\mathcal{X})} \le C \frac{1}{V_1(x_1)} \|h\|_{\mathcal{G}(\varrho,\vartheta)} \|g\|_{\mathrm{BMO}^+(\mathcal{X})},$$

*here and hereafter, for a fixed $x_1 \in \mathcal{X}$ and all $g \in \mathrm{BMO}\,(\mathcal{X})$,*

$$\|g\|_{\mathrm{BMO}^+(\mathcal{X})} := \|g\|_{\mathrm{BMO}\,(\mathcal{X})} + \frac{1}{V_1(x_1)} \int_{B(x_1,\,1)} |g(x)|\, d\mu(x).$$

We also need to explain the meaning of the product $f \times g$ for every $f \in H^1_{\mathrm{at}}(\mathcal{X})$ and $g \in \mathrm{BMO}\,(\mathcal{X})$ (see [49]). For any $h \in \mathcal{G}_0^\epsilon(\varrho,\vartheta)$, let

$$\langle f \times g, h \rangle := \langle gh, f \rangle := \int_{\mathcal{X}} g(x)h(x)f(x)\, d\mu(x).$$

From Lemma 4.8, it follows that $gh \in \mathrm{BMO}\,(\mathcal{X})$ and hence the above definition is well defined in the sense of the duality between $H^1_{\mathrm{at}}(\mathcal{X})$ and $\mathrm{BMO}\,(\mathcal{X})$.

Now we state the main result in [27] as follows, which is an extension of Theorem 2.3 from Euclidean spaces to metric measure spaces of homogeneous type.

**Theorem 4.9** ([27])**.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then there exist two bounded bilinear operators $\mathscr{L}: H^1_{\rm at}(\mathcal{X}) \times {\rm BMO}\,(\mathcal{X}) \to L^1(\mathcal{X})$ and $\mathscr{H}: H^1_{\rm at}(\mathcal{X}) \times {\rm BMO}\,(\mathcal{X}) \to H^{\log}(\mathcal{X})$, and a positive constant $C$ such that, for all $f \in H^1_{\rm at}(\mathcal{X})$ and $g \in {\rm BMO}\,(\mathcal{X})$,*

$$f \times g = \mathscr{L}(f,g) + \mathscr{H}(f,g) \quad in \quad (\mathcal{G}^{\epsilon}_0(\varrho, \vartheta))',$$

*where $\epsilon \in (0,1]$ and $\varrho$, $\vartheta \in (0, \epsilon]$, and*

$$\|\mathscr{L}(f,\, g)\|_{L^1(\mathcal{X})} + \|\mathscr{H}(f,\, g)\|_{H^{\log}(\mathcal{X})} \le C\|f\|_{H^1_{\rm at}(\mathcal{X})}\|g\|_{{\rm BMO}^+(\mathcal{X})}.$$

4.2. **Bilinear decompositions for the products of functions in $H^1_\rho(\mathcal{X})$ and ${\rm BMO}_\rho(\mathcal{X})$.** Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. In this subsection, we summarize the bilinear decompositions of products of functions in $H^1_\rho(\mathcal{X})$ and ${\rm BMO}_\rho(\mathcal{X})$ associated to the admissible function $\rho$.

We first recall the notion of approximations of the identity on RD-spaces from [36].

**Definition 4.10.** Let $\epsilon_1 \in (0, 1]$ and $\epsilon_2$, $\epsilon_3 \in (0,\infty)$. A family $\{S_k\}_{k\in\mathbb{Z}}$ of linear operators, which are bounded on $L^2(\mathcal{X})$, is called an *approximation of the identity of order* $(\epsilon_1, \epsilon_2, \epsilon_3)$ [for short, $(\epsilon_1, \epsilon_2, \epsilon_3)$-AI] if there exists a positive constant $C_0$ such that, for all $k \in \mathbb{Z}$ and $x$, $\widetilde{x}$, $y$, $\widetilde{y} \in \mathcal{X}$, $S_k(x, y)$, the integral kernel of $S_k$, is a measurable function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

(i) $|S_k(x, y)| \le C_0 \frac{1}{V_{2^{-k}}(x)+V(x,y)}[\frac{2^{-k}}{2^{-k}+d(x,y)}]^{\epsilon_2}$;

(ii) for $d(x, \widetilde{x}) \le [2^{-k} + d(x, y)]/2$,

$$|S_k(x, y) - S_k(\widetilde{x}, y)| \le C_0 \left[\frac{d(x,\widetilde{x})}{2^{-k} + d(x, y)}\right]^{\epsilon_1} \frac{1}{V_{2^{-k}}(x) + V(x, y)} \left[\frac{2^{-k}}{2^{-k} + d(x, y)}\right]^{\epsilon_2};$$

(iii) property (ii) also holds true with $x$ and $y$ interchanged;

(iv) for $d(x, \widetilde{x}) \le [2^{-k} + d(x, y)]/3$ and $d(y, \widetilde{y}) \le [2^{-k} + d(x, y)]/3$,

$$\begin{aligned} &|[S_k(x, y) - S_k(x, \widetilde{y})] - [S_k(\widetilde{x}, y) - S_k(\widetilde{x}, \widetilde{y})]| \\ &\quad \le C_0 \left[\frac{d(x,\widetilde{x})}{2^{-k} + d(x, y)}\right]^{\epsilon_1} \left[\frac{d(y,\widetilde{y})}{2^{-k} + d(x, y)}\right]^{\epsilon_1} \frac{1}{V_{2^{-k}}(x) + V(x, y)} \left[\frac{2^{-k}}{2^{-k} + d(x, y)}\right]^{\epsilon_3}; \end{aligned}$$

(v) $\int_{\mathcal{X}} S_k(x, z)\, d\mu(z) = 1 = \int_{\mathcal{X}} S_k(z, x)\, d\mu(z)$.

**Remark 4.11.** A sequence $\{\widetilde{S}_t\}_{t\in(0,\infty)}$ of bounded linear operators on $L^2(\mathcal{X})$ is called a *continuous approximation of the identity of order* $(\epsilon_1, \epsilon_2, \epsilon_3)$ [for short, $(\epsilon_1, \epsilon_2, \epsilon_3)$-CAI] if it satisfies (i) through (v) of Definition 4.10 with $2^{-k}$ replaced by $t$. It was shown by [67, Remark 2.2(ii)] that, if $\{S_k\}_{k\in\mathbb{Z}}$ is an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AI and, for any $t \in (2^{-k-1}, 2^{-k}]$ with $k \in \mathbb{Z}$, let $\widetilde{S}_t := S_k$, then $\{\widetilde{S}_t\}_{t\in(0,\infty)}$ is an $(\epsilon_1, \epsilon_2, \epsilon_3)$-CAI.

Then we recall the notion of admissible functions from [67].

**Definition 4.12.** A positive function $\rho$ is said to be *admissible* if there exist positive constants $C_2$ and $k_0$ such that, for all $x$, $y \in \mathcal{X}$,

$$\rho(y) \le C_2[\rho(x)]^{1/(1+k_0)}[\rho(x) + d(x, y)]^{k_0/(1+k_0)}.$$

**Remark 4.13.** It is obvious that constant functions are admissible functions with $C_2 := 1 =: k_0$. There exists a non-trivial class of admissible functions induced by the well-known reverse Hölder class $\mathcal{B}_q(\mathcal{X})$; see [67, p. 1201] for the details.

We also need the following assumption from [28] that there exists a specific generalized approximation of the identity on $(\mathcal{X}, d, \mu)$.

**Assumption 4.14.** There exists a family $\{T_t\}_{t\in(0,\infty)}$ of linear operators bounded on $L^2(\mathcal{X})$ with integrable kernels, still denoted by $\{T_t\}_{t\in(0,\infty)}$, satisfying that there exists an $(\epsilon_1, \epsilon_2, \epsilon_3)$-CAI $\{\widetilde{T}_t\}_{t\in(0,\infty)}$ (with integral kernels, still denoted by $\{\widetilde{T}_t\}_{t\in(0,\infty)}$) for some $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (1, \infty)$ and $\epsilon_3 \in (0, \infty)$, and positive constants $C$, $\delta_2 \in (1, \epsilon_2]$ and $\delta_1$, $\delta_3$ such that, for all $t \in (0, \infty)$ and $x, y \in \mathcal{X}$,

(i) $|T_t(x, y)| \le C \frac{1}{V_t(x)+V(x,y)} [\frac{t}{t+d(x,y)}]^{\delta_2} [\frac{\rho(x)}{t+\rho(x)}]^{\delta_3}$;

(ii) $|T_t(x, y) - \widetilde{T}_t(x, y)| \le C[\frac{t}{t+\rho(x)}]^{\delta_1} \frac{1}{V_t(x)+V(x,y)} [\frac{t}{t+d(x,y)}]^{\delta_2}$;

(iii) for any $N \in (0, \infty)$ large enough, there exists a positive constant $C_{(N)}$, depending on $N$, such that, for all $t \in (0, \infty)$ and $x, y \in \mathcal{X}$,

(iii)$_1$ if $[d(x, y)]^2 \ge t$, then

$$\left|\widetilde{T}_t(x, y)\right| \le C_{(N)} \frac{1}{V(x, y)} \left\{\frac{t}{[d(x, y)]^2}\right\}^N;$$

(iii)$_2$ if $[d(x, y)]^2 < t$, then

$$\left|\widetilde{T}_t(x, y)\right| \le C_{(N)} \frac{1}{V_{\sqrt{t}}(x)};$$

(iv) $|T_t(x, y)| \le C|\widetilde{T}_t(x, y)|$.

**Remark 4.15.** It was shown in [28, Remark 1.7] that there exists a $(1, N, N)$-CAI satisfying Assumption 4.14.

The following notions of maximal functions are from [67, Definition 2.5]. In what follows, for any numbers $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$.

**Definition 4.16.** Let $\epsilon_1 \in (0, 1]$, $\epsilon_2, \epsilon_3 \in (0, \infty)$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AI. Let $\rho$ be an admissible function as in Definition 4.12. For any $\varrho, \vartheta \in (0, \epsilon)$, $f \in (\mathcal{G}_0^\epsilon(\varrho, \vartheta))'$ and $x \in \mathcal{X}$, the *grand maximal function, associated to* $\rho$, $G_\rho^{\epsilon,\varrho,\vartheta}(f)$ is defined by setting

$$G_\rho^{\epsilon,\varrho,\vartheta}(f)(x) := \sup\{|\langle f, h\rangle| : h \in \mathcal{G}_0^\epsilon(\varrho, \vartheta), \|h\|_{\mathcal{G}(x,r,\varrho,\vartheta)} \le 1 \text{ for some } r \in (0, \rho(x))\}.$$

**Remark 4.17.** If there exists no ambiguity, then $G_\rho^{\epsilon,\varrho,\vartheta}$ is simply denoted by $G_\rho$.

Now we recall the notions of the Hardy space and its local version from [67, Definition 2.6],

**Definition 4.18.** Let $\epsilon \in (0, 1)$, $\varrho, \vartheta \in (0, \epsilon)$ and $\rho$ be an admissible function as in Definition 4.12.

(i) The *Hardy space* $H^1(\mathcal{X})$ is defined by setting

$$H^1(\mathcal{X}) := \left\{f \in (\mathcal{G}_0^\epsilon(\varrho,\vartheta))' : \ \|f\|_{H^1(\mathcal{X})} := \|\mathcal{M}(f)\|_{L^1(\mathcal{X})} < \infty\right\},$$

where $\mathcal{M}(f)$ is as in (4.2).

(ii) The *Hardy space* $H^1_\rho(\mathcal{X})$ *associated to* $\rho$ is defined by setting

$$H^1_\rho(\mathcal{X}) := \left\{f \in (\mathcal{G}_0^\epsilon(\varrho,\vartheta))' : \ \|f\|_{H^1_\rho(\mathcal{X})} := \|G_\rho(f)\|_{L^1(\mathcal{X})} < \infty\right\}.$$

The following notion of the local version of the space BMO$\,(\mathcal{X})$ is from [68, Definition 3.1].

**Definition 4.19.** Let $\rho$ be an admissible function as in Definition 4.12, $q \in (1,\infty]$ and

$$\mathcal{D} := \{B(x,r) \subset \mathcal{X} : \ x \in \mathcal{X}, \ r \geq \rho(x)\}.$$

A function $g \in L^q_{\rm loc}(\mathcal{X})$ is said to belong to the *space* $\mathrm{BMO}^q_\rho(\mathcal{X})$ if

$$\|g\|_{\mathrm{BMO}^q_\rho(\mathcal{X})} := \sup_{B\notin\mathcal{D}} \left\{\frac{1}{\mu(B)}\int_B |g(x)-m_B(g)|^q\,d\mu(x)\right\}^{1/q} + \sup_{B\in\mathcal{D}} \left\{\frac{1}{\mu(B)}\int_B |g(x)|^q\,d\mu(x)\right\}^{1/q} < \infty.$$

**Remark 4.20.** (i) By [68, Lemma 3.2], we know that $\mathrm{BMO}^q_\rho(\mathcal{X})$ with $q \in (1,\infty)$ coincides with $\mathrm{BMO}^1_\rho(\mathcal{X})$. In what follows, we denote $\mathrm{BMO}^1_\rho(\mathcal{X})$ simply by $\mathrm{BMO}_\rho(\mathcal{X})$.

(ii) Obviously, $\mathrm{BMO}_\rho(\mathcal{X}) \subset \mathrm{BMO}\,(\mathcal{X})$.

(iii) By [69, Theorem 2.1] and [67, Theorem 2.1], we know that the dual space of $H^1_\rho(\mathcal{X})$ is $\mathrm{BMO}_\rho(\mathcal{X})$.

In order to state the main result reviewed in this subsection, we need to illustrate the meaning of the product $f \times g$ for every $f \in H^1_\rho(\mathcal{X})$ and $g \in \mathrm{BMO}_\rho(\mathcal{X})$ (see [49]). For any $h \in \mathcal{G}_0^\epsilon(\varrho,\vartheta)$, let

$$\langle f\times g, h\rangle := \langle gh, f\rangle := \int_{\mathcal{X}} [g(x)h(x)]f(x)\,d\mu(x).$$

By [49, Proposition 4.1], we know that $gh \in \mathrm{BMO}_\rho(\mathcal{X})$ and hence the above definition is well defined in the sense of the duality between $H^1_\rho(\mathcal{X})$ and $\mathrm{BMO}_\rho(\mathcal{X})$.

The following result is just [28, Theorem 1.14], which is an extension of Theorem 2.15 from $\mathbb{R}^D$ to an RD-space and also a local version of Theorem 1.2.

**Theorem 4.21** ([28])**.** *Let $(\mathcal{X}, d, \mu)$ be an* RD*-space satisfying the additional Assumption 4.14 and $\rho$ an admissible function as in Definition 4.12. Then there exist two bounded bilinear operators $\mathscr{L}_\rho : H^1_\rho(\mathcal{X}) \times \mathrm{BMO}_\rho(\mathcal{X}) \to L^1(\mathcal{X})$ and $\mathscr{H}_\rho : H^1_\rho(\mathcal{X}) \times \mathrm{BMO}_\rho(\mathcal{X}) \to H^{\log}(\mathcal{X})$, and a positive constant $C$ such that, for all $f \in H^1_\rho(\mathcal{X})$ and $g \in \mathrm{BMO}_\rho(\mathcal{X})$,*

$$f\times g = \mathscr{L}_\rho(f,g) + \mathscr{H}_\rho(f,g) \quad \text{in} \quad (\mathcal{G}_0^\epsilon(\varrho,\vartheta))',$$

*where $\epsilon \in (0,1)$ and $\varrho$, $\vartheta \in (0,\epsilon]$, and*

$$\|\mathscr{L}_\rho(f,\, g)\|_{L^1(\mathcal{X})} + \|\mathscr{H}_\rho(f,\, g)\|_{H^{\log}(\mathcal{X})} \leq C\|f\|_{H^1_\rho(\mathcal{X})}\|g\|_{\mathrm{BMO}_\rho(\mathcal{X})}.$$

**Remark 4.22.** (i) If $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)$ is the Euclidean space, then Theorem 4.21 returns to Theorem 2.15.

(ii) Let $(\mathcal{X}, d, \mu)$ be an RD-space satisfying the additional Assumption 4.14. Then, by [28, Remark 1.15], Theorem 4.21 essentially improves [49, Theorem 1.2].

4.3. **Endpoint boundedness for commutators of singular integrals.** This subsection is devoted to reviewing the applications of the bilinear decompositions in Subsection 4.1 to the endpoint boundedness of commutators.

We first recall some notions and notation from [13]; see also [2, 16]. Let $C_b^s(\mathcal{X})$ be the space of all functions with bounded supports and the Hölder regularity $s$, where $s \in (0, \eta]$ is arbitrary and $\eta$ is as in Theorem 4.33 below. By [2, Proposition 4.5], we know that $C_b^s(\mathcal{X})$ is dense in $L^2(\mathcal{X})$. The dual space of $C_b^s(\mathcal{X})$ is denoted by $(C_b^s(\mathcal{X}))'$.

Now we recall the notion of Calderón-Zygmund operators from [13]; see also [2,16].

**Definition 4.23.** A function $K \in L^1_{\rm loc}(\{\mathcal{X} \times \mathcal{X}\} \setminus \{(x, x) : x \in \mathcal{X}\})$ is called a *Calderón-Zygmund kernel* if there exists a positive constant $C_{(K)}$, depending on $K$, such that

(i) for all $x$, $y \in \mathcal{X}$ with $x \neq y$,

$$|K(x,y)| \le C_{(K)} \frac{1}{V(x,y)}; \tag{4.3}$$

(ii) there exist positive constants $s \in (0, 1]$ and $c_{(K)}$, depending on $K$, such that

(ii)$_1$ for all $x$, $\widetilde{x}$, $y \in \mathcal{X}$ with $d(x,y) \ge c_{(K)} d(x, \widetilde{x}) > 0$,

$$|K(x,y) - K(\widetilde{x}, y)| \le C_{(K)} \left[\frac{d(x,\widetilde{x})}{d(x,y)}\right]^s \frac{1}{V(x,y)}; \tag{4.4}$$

(ii)$_2$ for all $x$, $y$, $\widetilde{y} \in \mathcal{X}$ with $d(x,y) \ge c_{(K)} d(y, \widetilde{y}) > 0$,

$$|K(x,y) - K(x, \widetilde{y})| \le C_{(K)} \left[\frac{d(y,\widetilde{y})}{d(x,y)}\right]^s \frac{1}{V(x,y)}. \tag{4.5}$$

Let $T : C_b^s(\mathcal{X}) \to (C_b^s(\mathcal{X}))'$ be a linear continuous operator. Then $T$ is called a *Calderón-Zygmund operator* with kernel $K$ satisfying (4.3), (4.4) and (4.5) if, for all $f \in C_b^s(\mathcal{X})$,

$$Tf(x) := \int_{\mathcal{X}} K(x,y) f(y)\, d\mu(y), \quad \forall\, x \notin \operatorname{supp}(f).$$

Now we introduce the notion of the space $H^1_g(\mathcal{X})$, which is a variant of [46, Definition 2.2].

**Definition 4.24.** Let $g$ be a non-constant $\mathrm{BMO}(\mathcal{X})$-function. A function $f$ in $H^1_{\rm at}(\mathcal{X})$ is said to belong to the *space* $H^1_g(\mathcal{X})$ if $[g, \mathcal{M}](f)$, defined by setting

$$[g, \mathcal{M}](f)(x) := \mathcal{M}\left(g(x) f(\cdot) - g(\cdot) f(\cdot)\right)(x), \quad \forall\, x \in \mathcal{X},$$

belongs to $L^1(\mathcal{X})$, where $\mathcal{M}$ is as in (4.2). Moreover, the *norm* of $f$ in $H^1_g(\mathcal{X})$ is defined by setting

$$\|f\|_{H^1_g(\mathcal{X})} := \|f\|_{H^1_{\rm at}(\mathcal{X})} \|g\|_{\mathrm{BMO}(\mathcal{X})} + \|[g, \mathcal{M}](f)\|_{L^1(\mathcal{X})}.$$

Here is the result on the endpoint boundedness of commutators, which is an extension of [46, Theorem 1.3]. Recall from [46] that the *symbol* $\mathcal{K}$ denotes the set of all sublinear operators $T$ satisfying

(i) $T$ is bounded from $H^1_{\rm at}(\mathcal{X})$ into $L^1(\mathcal{X})$ and from $L^1(\mathcal{X})$ into $L^{1,\infty}(\mathcal{X})$;
(ii) there exists a positive constant $C$ such that, for all $g \in {\rm BMO}\,(\mathcal{X})$ and $(1,2)$-atoms $a$ related to some balls $B \subset \mathcal{X}$,
$$\|[g - m_B(g)]\,Ta\|_{L^1(\mathcal{X})} \le C\|g\|_{{\rm BMO}\,(\mathcal{X})}.$$

**Theorem 4.25** ([46])**.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type and $g$ a non-constant* BMO $(\mathcal{X})$*-function. Then, for any $T \in \mathcal{K}$, the commutator $[g,T]$ is bounded from $H^1_g(\mathcal{X})$ into $L^1(\mathcal{X})$. In particular, if $T$ is a Calderón-Zygmund operator, then $[g,T]$ is bounded from $H^1_g(\mathcal{X})$ into $L^1(\mathcal{X})$ and, moreover, there exists a positive constant $C$ such that, for all $f \in H^1_g(\mathcal{X})$,*
$$\|[g,T](f)\|_{L^1(\mathcal{X})} \le C\|f\|_{H^1_g(\mathcal{X})}.$$

4.4. **A new proof of Theorem 4.9.** The goal of this subsection is to present a new proof of Theorem 4.9. To this end, we only need to give revised versions of [27, Lemma 3.7, Theorems 4.10 and 4.16 and Propositions 3.4 and 3.5]; see Lemmas 4.43, 4.37 and 4.44, and Propositions 4.38 and 4.39 below, respectively.

We first recall the notion of the geometrically doubling condition. Coifman and Weiss [13, pp. 66-68] indicated that spaces of homogeneous type are geometrically doubling. Recall that a metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x,r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most $N_0$.

**Remark 4.26.** It was shown by Hytönen [39] that a metric space $(\mathcal{X}, d)$ is geometrically doubling if and only if one of the following statements holds true:

(i) For any $\varepsilon \in (0,1)$ and any ball $B(x,r) \subset \mathcal{X}$, with $x \in \mathcal{X}$ and $r \in (0,\infty)$, there exists a finite ball covering $\{B(x_i, \varepsilon r)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most $N_0\varepsilon^{-n_0}$, where $N_0$ is the constant appearing in the definition of the geometrically doubling property and $n_0 := \log_2 N_0$;
(ii) For every $\varepsilon \in (0,1)$, any ball $B(x,r) \subset \mathcal{X}$, with $x \in \mathcal{X}$ and $r \in (0,\infty)$, contains at most $N_0\varepsilon^{-n_0}$ centers of disjoint balls $\{B(x_i, \varepsilon r)\}_i$;
(iii) There exists $M \in \mathbb{N}$ such that any ball $B(x,r) \subset \mathcal{X}$, with $x \in \mathcal{X}$ and $r \in (0,\infty)$, contains at most $M$ centers $\{x_i\}_{i=1}^M$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

We then present some notions, notation and conclusions from [26,27]. Let $(\mathcal{X}, d)$ be geometrically doubling. For every $k \in \mathbb{Z}$, a set of *reference dyadic points*, $\{x^k_\alpha\}_{\alpha\in\mathscr{A}_k}$, here and hereafter,

$$\mathscr{A}_k \text{ denotes some countable index set for each } k \in \mathbb{Z}, \tag{4.6}$$

is chosen as follows, where the Zorn lemma is applied (see, for example, [61, Theroem I.2]) since we consider the maximality. Let $\delta$ be a fixed small positive parameter. For example, it suffices to take $\delta \le \frac{1}{1000}$. For $k = 0$, let $\mathscr{X}^0 := \{x^0_\alpha\}_{\alpha\in\mathscr{A}_k}$ be a maximal collection of 1-separated points. Inductively, for any $k \in \mathbb{N}$, let

$$\mathscr{X}^k := \{x^k_\alpha\}_{\alpha\in\mathscr{A}_k} \supset \mathscr{X}^{k-1} \quad \text{and} \quad \mathscr{X}^{-k} := \{x^{-k}_\alpha\}_{\alpha\in\mathscr{A}_k} \subset \mathscr{X}^{-(k-1)} \tag{4.7}$$

be, respectively, maximal $\delta^k$-separated and $\delta^{-k}$-separated collections in $\mathcal{X}$ and in $\mathscr{X}^{-(k-1)}$. By [2, Lemma 2.1], we know that

$$d(x_\alpha^k, x_\beta^k) \geq \delta^k, \ \forall\, \alpha,\, \beta \in \mathscr{A}_k \text{ and } \alpha \neq \beta, \quad d(x, \mathscr{X}^k) := \inf_{\alpha \in \mathscr{A}_k} d(x, x_\alpha^k) < 2\delta^k. \tag{4.8}$$

Observe that the reference dyadic points $\{x_\alpha^k\}_{k\in\mathbb{Z},\, \alpha\in\mathscr{A}_k}$ satisfy [40, (2.3) and (2.4)] (with $c_0 = 1$ and $C_0 = 2$ therein), which further induces a system of dyadic cubes over geometrically doubling metric spaces as in [40, Theorem 2.2], which was re-formulated in [27, Theorem 2.3].

**Theorem 4.27** ([40]). *Let $(\mathcal{X}, d)$ be a metric space satisfying the geometrically doubling condition. For any $k \in \mathbb{Z}$, let $\mathscr{A}_k$ be as in (4.6). Then there exist families of sets, $\mathring{Q}_\alpha^k \subset Q_\alpha^k \subset \overline{Q}_\alpha^k$, which are said to be, respectively,* open, half-open *and* closed dyadic cubes*, such that:*

- (i) *$\mathring{Q}_\alpha^k$ and $\overline{Q}_\alpha^k$ represent the* interior *and the* closure *of $Q_\alpha^k$, respectively;*
- (ii) *if $\ell \in \mathbb{Z} \cap [k, \infty)$ and $\alpha,\, \beta \in \mathscr{A}_k$, then $Q_\beta^\ell \subset Q_\alpha^k$ and $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ holds true alternatively;*
- (iii) *$\mathcal{X} = \bigcup_{\alpha\in\mathscr{A}_k} Q_\alpha^k$ (disjoint union);*
- (iv) *for all $\alpha \in \mathscr{A}_k$, $B\left(x_\alpha^k, \frac{1}{3}\delta^k\right) \subset Q_\alpha^k \subset B\left(x_\alpha^k, 4\delta^k\right) =: B\left(Q_\alpha^k\right)$;*
- (v) *if $\ell \in \mathbb{Z} \cap [k, \infty)$, $\alpha,\, \beta \in \mathscr{A}_k$ and $Q_\beta^\ell \subset Q_\alpha^k$, then $B(Q_\beta^\ell) \subset B(Q_\alpha^k)$.*

*The open and closed cubes $\mathring{Q}_\alpha^k$ and $\overline{Q}_\alpha^k$, with $(k, \alpha) \in \mathscr{A}$, here and hereafter,*

$$\mathscr{A} := \{(k, \alpha) : \ k \in \mathbb{Z}, \ \alpha \in \mathscr{A}_k\}, \tag{4.9}$$

*depend only on the points $x_\beta^\ell$ for $\beta \in \mathscr{A}_\ell$ and $\ell \in \mathbb{Z} \cap [k, \infty)$. The half-open cubes $Q_\alpha^k$, with $(k, \alpha) \in \mathscr{A}$, depend on $x_\beta^\ell$ for $\beta \in \mathscr{A}_\ell$ and $\ell \in \mathbb{Z} \cap [\min\{k, k_0\}, \infty)$, where $k_0 \in \mathbb{Z}$ is a preassigned number in the construction.*

In what follows, for any set $E$, we use $\#E$ to denote its *cardinality* (the number of its elements).

**Remark 4.28.** By [26, Remark 2.4(ii)], we know that, for any $k \in \mathbb{Z}$ and $\alpha \in \mathscr{A}_k$, there exists a set $L(k, \alpha) \subset \mathscr{A}_{k+1}$ with $1 \leq \#L(k, \alpha) \leq \widetilde{N}_0$ such that

$$Q_\alpha^k = \bigcup_{\beta \in L(k,\, \alpha)} Q_\beta^{k+1}, \tag{4.10}$$

where $\widetilde{N}_0 \in \mathbb{N}$ is a constant independent of $k$ and $\alpha$.

We also need the following useful estimate about the 1-separated set from [2, Lemma 6.4].

**Lemma 4.29** ([2]). *Let $\Xi$ be a 1-separated set in a geometrically doubling metric space $(\mathcal{X}, d)$ with positive constant $N_0$. Then, for any $\epsilon \in (0, \infty)$, there exists a positive constant $C_{(\epsilon,\, N_0)}$, depending on $\epsilon$ and $N_0$, such that*

$$\sup_{a\in\mathcal{X}} e^{\epsilon d(a,\, \Xi)/2} \sum_{b\in\Xi} e^{-\epsilon d(a,\, b)} \leq C_{(\epsilon,\, N_0)},$$

*here and hereafter, for any set $\Xi \subset \mathcal{X}$ and $x \in \mathcal{X}$, $d(x, \Xi) := \inf_{a\in\Xi} d(x, a)$.*

Before recalling the orthonormal basis of regular wavelets, let $(\Omega, \mathscr{F}, \mathbb{P}_\omega)$ be the *natural product probability measure space* as in [2], where $\mathscr{F}$ represents the smallest $\sigma$-algebra containing the set

$$\left\{\prod_{k\in\mathbb{Z}} A_k :\ A_k \subset \Omega_k := \{0,1,\ldots,L\}\times\{1,\ldots,M\} \text{ and only finite many } A_k \neq \Omega_k\right\},$$

with $L$ and $M$ are the same as in [2, p. 270]. For every $(k,\alpha) \in \mathscr{A}$ with $\mathscr{A}$ as in (4.9), the spline function is defined by setting, for any $x \in \mathcal{X}$,

$$s_\alpha^k(x) := \mathbb{P}_\omega\left(\left\{\omega \in \Omega :\ x \in \overline{Q}_\alpha^k(\omega)\right\}\right).$$

Then the splines have the following properties:

(i) for each $(k,\alpha) \in \mathscr{A}$ and $x \in \mathcal{X}$,

$$\chi_{B(x_\alpha^k,\, \frac{4}{\delta}{}^k)}(x) \le s_\alpha^k(x) \le \chi_{B(x_\alpha^k,\, 8\delta^k)}(x); \tag{4.11}$$

(ii) for every $k \in \mathbb{Z}$, $\alpha$, $\beta \in \mathscr{A}_k$ (with $\mathscr{A}_k$ as in (4.6)) and $x \in \mathcal{X}$,

$$s_\alpha^k(x_\beta^k) = \delta_{\alpha\beta}, \quad s_\alpha^k(x) = \sum_{\beta\in\mathscr{T}_{k+1}} p_{\alpha\beta}^k s_\beta^{k+1}(x) \quad \text{and} \quad \sum_{\alpha\in\mathscr{A}_k} s_\alpha^k(x) = 1,$$

where, for any $k \in \mathbb{Z}$, $\mathscr{T}_{k+1} \subset \mathscr{A}_{k+1}$ denotes some countable index set,

$$\delta_{\alpha\beta} := \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

and $\{p_{\alpha\beta}^k\}_{\beta\in\mathscr{T}_{k+1}} \subset [0,1]$ is a set of numbers with finite nonzero items;

(iii) there exist positive constants $C$ and $\eta \in (0,1]$ such that, for all $(k,\alpha) \in \mathscr{A}$ and $x$, $y \in \mathcal{X}$,

$$\left|s_\alpha^k(x) - s_\alpha^k(y)\right| \le C\left[\frac{d(x,y)}{\delta^k}\right]^\eta, \tag{4.12}$$

where the constants $\eta$ and $C$ are independent of the choices of $k$ and $\alpha$.

Now we are ready to recall the orthonormal basis of regular wavelets constructed by Auscher and Hytönen ( [2, Theorem 7.1]); see also [26] for an equivalent version with some small modifications on the notation

$$\{\psi_{\alpha,\beta}^k\}_{(k,\alpha)\in\widetilde{\mathscr{A}},\, \beta\in\widetilde{L}(k,\alpha)} := \{\psi_\beta^k\}_{k\in\mathbb{Z},\, \beta\in\mathscr{G}_k}, \tag{4.13}$$

where $\mathscr{G}_k := \{\beta \in \mathscr{A}_{k+1} :\ x_\beta^{k+1} \notin \mathscr{X}^k\}$,

$$\widetilde{\mathscr{A}} := \{(k,\alpha) \in \mathscr{A} :\ \#L(k,\alpha) > 1\} \tag{4.14}$$

and, for all $(k,\alpha) \in \widetilde{\mathscr{A}}$,

$$\widetilde{L}(k,\alpha) := L(k,\alpha) \setminus \left\{\beta \in L(k,\alpha) :\ x_\beta^{k+1} = x_\alpha^k\right\} \tag{4.15}$$

with $L(k,\alpha)$ as in (4.10).

From [2, Theorem 5.1], it follows that there exists a linear, bounded (uniformly on $k \in \mathbb{Z}$), and injective map $U_k : \ell^2(\mathscr{A}_k) \to L^2(\mathcal{X})$ with closed range, defined by setting

$$U_k\lambda := \sum_{\alpha\in\mathscr{A}_k} \frac{\lambda_\alpha}{\sqrt{\mu_\alpha^k}} s_\alpha^k, \quad \forall\, \lambda := \left\{\lambda_\alpha^k\right\}_{\alpha\in\mathscr{A}_k} \in \ell^2(\mathscr{A}_k),$$

here and hereafter, for any $(k,\alpha) \in \mathscr{A}$,

$$\mu_\alpha^k := \mu(B(x_\alpha^k, \delta^k)) =: V(x_\alpha^k, \delta^k) \tag{4.16}$$

and

$$\ell^2(\mathscr{A}_k) := \left\{ \lambda := \{\lambda_\alpha^k\}_{\alpha\in\mathscr{A}_k} \subset \mathbb{C} : \ \|\lambda\|_{\ell^2(\mathscr{A}_k)} := \left[\sum_{\alpha\in\mathscr{A}_k} \left|\lambda_\alpha^k\right|^2\right]^{1/2} < \infty \right\}.$$

Observe that, if we let $k \in \mathbb{Z}$, $\lambda$, $\widetilde{\lambda} \in \ell^2(\mathscr{A}_k)$, $f := U_k\lambda$ and $\widetilde{f} := U_k\widetilde{\lambda}$, then

$$\left(f, \widetilde{f}\right)_{L^2(\mathcal{X})} = \left(M_k\lambda, \widetilde{\lambda}\right)_{\ell^2(\mathscr{A}_k)},$$

where $M_k$ is the infinite matrix which has entries

$$M_k(\alpha,\beta) := \frac{(s_\alpha^k, s_\beta^k)_{L^2(\mathcal{X})}}{\sqrt{\mu_\alpha^k \mu_\beta^k}}, \quad \forall\, \alpha,\ \beta \in \mathscr{A}_k.$$

For all $k \in \mathbb{Z}$, let $U_k^*$ denote the *adjoint operator* of $U_k$ and $V_k := U_k(\ell^2(\mathscr{A}_k))$. The following result from [2] implies that $\{V_k\}_{k\in\mathbb{Z}}$ is a *multiresolution analysis* (for short, MRA) of $L^2(\mathcal{X})$.

**Theorem 4.30** ([2]). *Suppose that $(\mathcal{X}, d, \mu)$ is a space of homogeneous type. Let $k \in \mathbb{Z}$ and $V_k$ be the closed linear span of $\{s_\alpha^k\}_{\alpha\in\mathscr{A}_k}$. Then $V_k \subset V_{k+1}$,*

$$\overline{\bigcup_{k\in\mathbb{Z}} V_k} = L^2(\mathcal{X}) \quad \text{and} \quad \bigcap_{k\in\mathbb{Z}} V_k = \{0\}.$$

*Moreover, the functions $\{s_\alpha^k/\sqrt{\mu_\alpha^k}\}_{\alpha\in\mathscr{A}_k}$ form a Riesz basis of $V_k$: for all sequences of complex numbers $\{\lambda_\alpha^k\}_{\alpha\in\mathscr{A}_k}$,*

$$\left\|\sum_{\alpha\in\mathscr{A}_k} \lambda_\alpha^k s_\alpha^k\right\|_{L^2(\mathcal{X})} \sim \left[\sum_{\alpha\in\mathscr{A}_k} \left|\lambda_\alpha^k\right|^2 \mu_\alpha^k\right]^{1/2}$$

*with equivalent positive constants independent of $k$ and $\{\lambda_\alpha^k\}_{\alpha\in\mathscr{A}_k}$, where $\mu_\alpha^k$ is as in* (4.16).

Now we recall an orthonormal basis $\{\phi_\alpha^k\}_{\alpha\in\mathscr{A}_k}$ in $V_k$ from [2, Theorem 6.1], where, for any $k \in \mathbb{Z}$, $\mathscr{A}_k$ is as in (4.6).

**Theorem 4.31** ([2]). *Let $k \in \mathbb{Z}$ and $(\mathcal{X}, d, \mu)$ be a space of homogeneous type. Then there exist a positive constant $\nu$ and an orthonormal basis $\{\phi_\alpha^k\}_{\alpha\in\mathscr{A}_k}$ in $V_k$ such that, for any $x \in \mathcal{X}$,*

$$\sqrt{\mu_\alpha^k}\left|\phi_\alpha^k(x)\right| \le Ce^{-\nu\delta^{-k} d(x_\alpha^k, x)}$$

*and, for any $x$, $y \in \mathcal{X}$ with $d(x,y) \le \delta^k$,*

$$\sqrt{\mu_\alpha^k}\left|\phi_\alpha^k(x) - \phi_\alpha^k(x)\right| \le C\left[\frac{d(x,y)}{\delta^k}\right]^\eta e^{-\nu\delta^{-k}d(x_\alpha^k,x)},$$

*where $\mu_\alpha^k$ is as in* (4.16) *and $\eta$ as in* (4.12).

**Remark 4.32.** We point out that there exists a gap in the proof of Theorem 4.9 that $\{s_\alpha^k/\sqrt{\mu_\alpha^k}\}_{\alpha\in\mathscr{A}_k}$ is not a orthogonal basis of $V_k$. Thus, the following representation is not correct:

$$f = \sum_{\alpha\in\mathscr{A}_k}\left(f, \frac{s_\alpha^k}{\mu_\alpha^k}\right)s_\alpha^k, \quad \forall\, f \in V_k. \tag{4.17}$$

Instead of (4.17), thanks to the orthonormal basis $\{\phi_\alpha^k\}_{\alpha\in\mathscr{A}_k}$, we can use the following representation:

$$f = \sum_{\alpha\in\mathscr{A}_k}\left(f, \phi_\alpha^k\right)\phi_\alpha^k, \quad \forall\, f \in V_k.$$

Now we state the remarkable orthonormal basis in [2]; see [26] for an equivalent version.

**Theorem 4.33** ([2])**.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then there exists an orthonormal basis $\{\psi_{\alpha,\beta}^k\}_{(k,\alpha)\in\widetilde{\mathscr{A}},\beta\in\widetilde{L}(k,\alpha)}$ of $L^2(\mathcal{X})$ and positive constants $C$, $\nu$, and $\eta \in (0,1]$ such that*

$$\left|\psi_{\alpha,\beta}^k(x)\right| \le \frac{C}{\sqrt{V(x_\beta^{k+1},\delta^k)}}e^{-\nu\delta^{-k}d(x_\beta^{k+1},x)}, \quad \forall\, x \in \mathcal{X}, \tag{4.18}$$

$$\left|\psi_{\alpha,\beta}^k(x) - \psi_{\alpha,\beta}^k(y)\right| \le \frac{C}{\sqrt{V(x_\beta^{k+1},\delta^k)}}\left[\frac{d(x,y)}{\delta^k}\right]^\eta e^{-\nu\delta^{-k}d(x_\beta^{k+1},x)}$$

*for all $x$, $y \in \mathcal{X}$ with $d(x,y) \le \delta^k$, and*

$$\int_{\mathcal{X}}\psi_{\alpha,\beta}^k(x)\,d\mu(x) = 0. \tag{4.19}$$

In what follows, we let

$$\mathscr{I} := \left\{(j,\alpha,\beta):\ (k,\alpha)\in\widetilde{\mathscr{A}},\ \beta\in\widetilde{L}(k,\alpha)\right\}, \tag{4.20}$$

where $\widetilde{\mathscr{A}}$ and $\widetilde{L}(k,\alpha)$ are, respectively, as in (4.14) and (4.15).

We now need to recall more notation from [26, 27]. Let

$$\mathscr{C} := \{(k,\beta):\ k\in\mathbb{Z},\ \beta\in\mathscr{G}_k\} \tag{4.21}$$

with $\mathscr{G}_k$ as in (4.13). We choose a fixed collection

$$\{\mathscr{C}_N:\ N\in\mathbb{N},\ \mathscr{C}_N\subset\mathscr{C} \text{ and } \mathscr{C}_{\mathrm{N}} \text{ is finite}\} \tag{4.22}$$

such that $\mathscr{C}_N \uparrow \mathscr{C}$, namely, for any $N \in \mathbb{N}$, $\mathscr{C}_N \subset \mathscr{C}_{N+1}$ and $\mathscr{C} = \bigcup_{N\in\mathbb{N}}\mathscr{C}_N$.

Let $N \in \mathbb{N}$, $g \in \mathrm{BMO}\,(\mathcal{X})$, $x_1$ and $r_1$ be as in Theorem 4.34,

$$g_N := \sum_{(j,\,\beta)\in\mathscr{G}_N} \left\langle g, \psi_\beta^j \right\rangle \psi_\beta^j,$$

$$\widetilde{g}_N := \sum_{(j,\,\beta)\in\mathscr{G}_N} \left\langle g, \psi_\beta^j \right\rangle \left[\psi_\beta^j - \chi_{\{k\in\mathbb{Z}:\ \delta^k>r_1\}}(j)\psi_\beta^j(x_1)\right]$$

and

$$c_{(N)} := \widetilde{g}_N - g_N.$$

We then recall the following wavelet characterization of $\mathrm{BMO}(\mathcal{X})$ from [2, Theorem 11.4]. A sequence $\{b_\beta^j\}_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j}$ is said to belong to the *Carleson sequence space* $\mathrm{Car}(\mathcal{X})$ if

$$\left\|\left\{b_\beta^j\right\}_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j}\right\|_{\mathrm{Car}(\mathcal{X})} := \sup_{k\in\mathbb{Z},\,\alpha\in\mathscr{A}_k}\left[\frac{1}{\mu(Q_\alpha^k)}\sum_{\substack{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j\\ (j+1,\,\beta)\le(k,\,\alpha)}}\left|b_\beta^j\right|^2\right]^{1/2} < \infty.$$

**Theorem 4.34** ([2])**.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then the space $\mathrm{BMO}\,(\mathcal{X})/\mathbb{C}$ ($\mathrm{BMO}(\mathcal{X})$ functions modulo constants) and Car($\mathcal{X}$) are isomorphic. The isomorphism is represented by $b \mapsto \{\langle b, \psi_\beta^j\rangle\}_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j} =: \{b_\beta^j\}_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j}$ with the inverse given by*

$$\{b_\beta^j\}_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j} \mapsto \sum_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j} b_\beta^j\left[\psi_\beta^j - \chi_{\{k\in\mathbb{Z}:\ \delta^k>r_1\}}(j)\psi_\beta^j(x_1)\right] =: \widetilde{b},$$

*where $\mathscr{G}_j$ with $j \in \mathbb{Z}$ is as in* (4.13)*, the series converges in $L^2_{\mathrm{loc}}\,(\mathcal{X})$ for every $x_1 \in \mathcal{X}$ and $r_1 \in (0,\infty)$, and the choices of $x_1$ and $r_1$ only alter the result by an additive constant.*

**Remark 4.35.** (i) From the proof of [2, Theorem 11.4], it follows that, if $b \in \mathrm{BMO}\,(\mathcal{X})$, then $\widetilde{b} - b =$ constant and hence

$$b = \widetilde{b} = \sum_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j} b_\beta^j\left[\psi_\beta^j - \chi_{\{k\in\mathbb{Z}:\ \delta^k>r_1\}}(j)\psi_\beta^j(x_1)\right]$$

converges in $\mathrm{BMO}\,(\mathcal{X})$ for every $(x_1, r_1) \in \mathcal{X}\times(0,\infty)$.

(ii) The proof of [2, Theorem 11.4] implies that there exists a positive constant $C$ such that, for all $b \in \mathrm{BMO}\,(\mathcal{X})$,

$$\left\|\left\{b_\beta^j\right\}_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j}\right\|_{\mathrm{Car}(\mathcal{X})} \le C\|b\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

(iii) Let $b \in \mathrm{BMO}\,(\mathcal{X})$ and $c_{(b)} := \sum_{j\in\mathbb{Z},\,\beta\in\mathscr{G}_j}\langle b, \psi_\beta^j\rangle\chi_{\{k\in\mathbb{Z}:\ \delta^k>r_1\}}(j)\psi_\beta^j(x_1)$. It was shown in [57] that $c_{(b)}$ is finite.

If two functions in $L^2(\mathcal{X})$ both have finite wavelet decompositions, we state the following conclusion from [27, Lemma 3.1].

**Lemma 4.36** ([27]). *Suppose that $(\mathcal{X}, d, \mu)$ is a metric measure space of homogeneous type. Let $f$, $g \in L^2(\mathcal{X})$, $\{V_k\}_{k\in\mathbb{Z}}$ be an MRA of $L^2(\mathcal{X})$ as in Theorem 4.30, $W_k$ be the orthogonal complement [in $L^2(\mathcal{X})$] of $V_k$ in $V_{k+1}$ and $P_k$ and $Q_k$ be the projection operators from $L^2(\mathcal{X})$, respectively, onto $V_k$ and $W_k$. Suppose that $f$ and $g$ both have finite wavelet decompositions, namely, there exist $M_1$, $M_2 \in \mathbb{N}$ such that*

$$f = \sum_{k=-M_1}^{M_1} \sum_{\beta\in\mathscr{G}_k} \left(f, \psi^k_\beta\right) \psi^k_\beta \quad \text{and} \quad g = \sum_{k=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_k} \left(g, \psi^k_\beta\right) \psi^k_\beta, \tag{4.23}$$

*where $\mathscr{G}_k$ for $k \in \mathbb{Z}$ is as in (4.13). Then*

$$\begin{aligned} fg &= \sum_{k\in\mathbb{Z}}(P_k f)(Q_k g) + \sum_{k\in\mathbb{Z}}(Q_k f)(P_k g) + \sum_{k\in\mathbb{Z}}(Q_k f)(Q_k g) \\ &=: \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g) \quad \text{in} \quad L^2(\mathcal{X}). \end{aligned} \tag{4.24}$$

The following lemma is a new version of [27, Theorems 4.10] with some slight modifications. We present some details here.

**Lemma 4.37.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then, for any $(1, 2)$-atom $a$ and $g \in \mathrm{BMO}\,(\mathcal{X})$, the series*

$$\sum_{j\in\mathbb{Z}} \sum_{\alpha\in\mathscr{A}_j} \sum_{\beta\in\mathscr{G}_j} \left(a, \phi^j_\alpha\right) \left(g, \psi^j_\beta\right) \phi^j_\alpha \psi^j_\gamma$$

*converges to some element in $H^1_{\rm at}(\mathcal{X})$, denoted by $\Pi_1(a, g)$ and*

$$\|\Pi_1(a, g)\|_{H^1_{\rm at}(\mathcal{X})} \le C\|g\|_{\mathrm{BMO}\,(\mathcal{X})},$$

*where $C$ is a positive constant independent of $a$ and $g$. Moreover, $\Pi_1$ can be extended to a bounded bilinear operator from $H^1_{\rm at}(\mathcal{X}) \times \mathrm{BMO}\,(\mathcal{X})$ into $H^1_{\rm at}(\mathcal{X})$.*

To show Lemma 4.37, we recall more results and notation from [27].

For every $k$, $j \in \mathbb{Z}$ and $(j, \beta) \in \mathscr{C}$, write

$$\mathscr{A}^k_{j,\beta} := \left\{\alpha \in \mathscr{A}_j :\ 2^k\delta^{j+1} \le d(x^j_\alpha, y^j_\beta) < 2^{k+1}\delta^{j+1}\right\}, \tag{4.25}$$

where $x^j_\alpha$ is as in (4.7) with $k$ replaced by $j$, and $y^j_\beta := x^{j+1}_\beta$ for $\beta \in \mathscr{G}_j$. From the geometrically doubling condition and Remark 4.26(ii), we deduce that, for all $j$, $k \in \mathbb{Z}$ and $\beta \in \mathscr{G}_j$,

$$M^k_{j,\beta} := \#\mathscr{A}^k_{j,\beta} \le N_0 2^{(k+1)G_0} =: m_k \tag{4.26}$$

with $G_0$ and $N_0$ same as in Remark 4.26(i).

We now relabel the set $\mathscr{A}^k_{j,\beta}$ as $\mathscr{A}^k_{j,\beta} =: \{\alpha^i_{j,\beta}\}_{i=1}^{M^k_{j,\beta}}$. If $M^k_{j,\beta} < m_k$, then we further enlarge $\mathscr{A}^k_{j,\beta}$ to $\{\alpha^i_{j,\beta}\}_{i=1}^{m_k}$ with $s^j_{\alpha^i_{j,\beta}} := 0$ for any $i \in \mathbb{N} \cap (M^k_{j,\beta}, m_k]$. If $M^k_{j,\beta} = m_k$, then the set $\mathscr{A}^k_{j,\beta}$ remains unchanged. Let $\alpha := \alpha^i_{j,\beta} \in \mathscr{A}^k_{j,\beta}$, $g \in L^2(\mathcal{X})$,

$$\widetilde{\psi}^{k,i}_{j,\beta} := e^{\nu\delta 2^{k-2}} \sqrt{\mu^j_\alpha}\phi^j_\alpha \psi^j_\beta \quad \text{and} \quad U^N_{k,i} g := \sum_{(j,\beta)\in\mathscr{C}_N} \left(g, \psi^j_\beta\right) \widetilde{\psi}^{k,i}_{j,\beta}, \tag{4.27}$$

where $\psi_\beta^j$ is as in Theorem 4.33 with $(k,\alpha)$ replaced by $(j,\beta)$. We also let

$$\mathscr{Y}^k := \mathscr{X}^{k+1} \setminus \mathscr{X}^k$$

with $\mathscr{X}^k$ as in (4.7). Then we know that $U_{k,i}^N g \in L^2(\mathcal{X})$ for all $g \in L^2(\mathcal{X})$, since $\mathscr{C}_N$ is finite. Moreover, we display the following result from [27, Proposition 3.4] with $s_\alpha^j$ replaced by $\sqrt{\mu_\alpha^j}\phi_\alpha^j$.

**Proposition 4.38.** *Suppose that $(\mathcal{X}, d, \mu)$ is a metric measure space of homogeneous type. Let $U_{k,i}^N$ be defined as in (4.27) for $N \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i \in \{1,\dots,m_k\}$ with $m_k$ as in (4.26). Then there exists a positive constant $C$, independent of $N$, $k$ and $i$, such that, for all $g$, $h \in L^2(\mathcal{X})$,*

$$\left|\left(U_{k,i}^N g, h\right)\right| \le C \left[\sum_{(j,\beta)\in\mathscr{C}_N} \left|\left(g, \psi_\beta^j\right)\right|^2\right]^{1/2} \|h\|_{L^2(\mathcal{X})} \le C\|g\|_{L^2(\mathcal{X})}\|h\|_{L^2(\mathcal{X})}. \tag{4.28}$$

*Proof.* By the proof of [27, Proposition 3.4], we conclude that, for all $g$, $h \in L^2(\mathcal{X})$,

$$\left|\left(U_{k,i}^N g, h\right)\right| \le \|g\|_{L^2(\mathcal{X})} \left[\sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j} \left|\left(\widetilde{\psi}_{j,\beta}^{k,i}, h\right)\right|^2\right]^{1/2}.$$

Thus, (4.28) is reduced to showing that

$$\mathrm{I} := \left[\sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j} \left|\left(\widetilde{\psi}_{j,\beta}^{k,i}, h\right)\right|^2\right]^{1/2} \lesssim \|h\|_{L^2(\mathcal{X})}. \tag{4.29}$$

To this end, similarly to the proof of [27, Proposition 3.4], we write

$$\begin{aligned}
\mathrm{I} &\le \left\{\sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j}\left[\sum_{s\in\mathbb{Z}}\sum_{\gamma\in\mathscr{G}_s}\left|\left(h,\psi_\gamma^s\right)\right|\left|\left(\psi_\gamma^s,\widetilde{\psi}_{j,\beta}^{k,i}\right)\right|\right]^2\right\}^{1/2}\\
&\le \left\{\sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j}\left|\left(h,\psi_\beta^j\right)\right|^2\left|\left(\psi_\beta^j,\widetilde{\psi}_{j,\beta}^{k,i}\right)\right|^2\right\}^{1/2}\\
&\quad+\left\{\sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j}\left[\sum_{\{\gamma\in\mathscr{G}_j:\ \gamma\neq\beta\}}\left|\left(h,\psi_\gamma^j\right)\right|\left|\left(\psi_\gamma^j,\widetilde{\psi}_{j,\beta}^{k,i}\right)\right|\right]^2\right\}^{1/2}\\
&\quad+\left\{\sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j}\left[\sum_{s=-\infty}^{j-1}\sum_{\gamma\in\mathscr{G}_s}\left|\left(h,\psi_\gamma^s\right)\right|\left|\left(\psi_\gamma^s,\widetilde{\psi}_{j,\beta}^{k,i}\right)\right|\right]^2\right\}^{1/2}\\
&\quad+\left\{\sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j}\left[\sum_{s=j+1}^{\infty}\sum_{\gamma\in\mathscr{G}_s}\cdots\right]^2\right\}^{1/2} =: \sum_{t=1}^4 \mathrm{I}_t.
\end{aligned}$$

As for $\mathrm{I}_1$, we first estimate $|(\psi_\beta^j, \widetilde{\psi}_{j,\beta}^{k,i})|$ for any $(j,\beta) \in \mathscr{C}$, where $\mathscr{C}$ is as in (4.21) and $\alpha \in \mathscr{A}_j$ with $\mathscr{A}_j$ as in (4.6).

Notice that, from (1.7), we deduce that, for any $r_0$, $\nu_0 \in (0,\infty)$ and $x_0 \in \mathcal{X}$,

$$\begin{aligned}
(4.30)\qquad &\int_{\mathcal{X}} e^{-\nu_0 d(x,x_0)/r_0}\, d\mu(x) \\
&\lesssim \int_{B(x_0,r_0)} e^{-\nu_0 d(x,x_0)/r_0}\, d\mu(x) + \sum_{\ell=1}^\infty \int_{B(x_0,(\ell+1)r_0)\setminus B(x_0,\ell r_0)} \cdots \\
&\lesssim V(x_0,r_0) + \sum_{\ell=1}^\infty e^{-\nu_0 \ell} V(x_0,[\ell+1]r_0) \\
&\lesssim V(x_0,r_0) + \sum_{\ell=1}^\infty e^{-\nu_0 \ell}(\ell+1)^n V(x_0,r_0) \lesssim V(x_0,r_0).
\end{aligned}$$

By (4.27), (4.18), $d(x_\alpha^j, y_\beta^j) \ge 2^k \delta^{j+1}$, (4.30) and (4.49), we conclude that

$$\begin{aligned}
\left|\left(\psi_\beta^j, \widetilde{\psi}_{j,\beta}^{k,i}\right)\right| &\le \int_{\mathcal{X}} \left|\psi_\beta^j(x)\widetilde{\psi}_{j,\beta}^{k,i}(x)\right| d\mu(x) \lesssim e^{\nu\delta 2^{k-2}} \sqrt{\mu_\alpha^j} \int_{\mathcal{X}} \left|\phi_\alpha^j(x)\right| \left|\psi_\beta^j(x)\right|^2 d\mu(x) \\
&\lesssim e^{\nu\delta 2^{k-2}} \frac{1}{V(y_\beta^j,\delta^j)} \int_{\mathcal{X}} e^{-2\nu\delta^{-j} d(y_\beta^j,x)} e^{-\nu\delta^{-j} d(x_\alpha^j,x)}\, d\mu(x) \\
&\lesssim e^{\nu\delta 2^{k-2}} e^{-\nu\delta^{-j} d(y_\beta^j,x_\alpha^j)} \frac{1}{V(y_\beta^j,\delta^j)} \int_{\mathcal{X}} e^{-\nu\delta^{-j} d(y_\beta^j,x)}\, d\mu(x) \\
&\lesssim e^{\nu\delta 2^{k-2}} e^{-\nu\delta 2^k} \lesssim 1,
\end{aligned}$$

which, together with Theorem 4.33, implies that

$$\mathrm{I}_1 \lesssim \left\{\sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\left(h, \psi_\beta^j\right)\right|^2\right\}^{1/2} \sim \|h\|_{L^2(\mathcal{X})}.$$

From the estimate of $\mathrm{I}_2$ in the proof of [27, Proposition 3.4], we deduce that

$$\begin{aligned}
\mathrm{I}_2 \lesssim &\sum_{s=0}^\infty 2^{\frac{(s+1)}{2} G_0} \\
&\times \left\{\sum_{j\in\mathbb{Z}} \sum_{\gamma\in\mathscr{G}_j} \left|(h,\psi_\gamma^j)\right|^2 \sum_{\{\beta\in\mathscr{G}_j:\ 2^s\delta^{j+1} \le d(y_\gamma^j, y_\beta^j) < 2^{s+1}\delta^{j+1}\}} \left|\left(\psi_\gamma^j, \widetilde{\psi}_{j,\beta}^{k,i}\right)\right|^2\right\}^{1/2}.
\end{aligned}$$

Now we estimate $|(\psi_\gamma^j, \widetilde{\psi}_{j,\beta}^{k,i})|$ for any $(j,\gamma) \in \mathscr{C}$ with $\mathscr{C}$ as in (4.21), $s \in \mathbb{Z}_+$ and $\beta$ satisfying $2^s\delta^{j+1} \le d(y_\gamma^j, y_\beta^j) < 2^{s+1}\delta^{j+1}$. By (4.11), (4.18), $\alpha \in \mathscr{A}_{j,\beta}^k$, $2^s\delta^{j+1} \le d(y_\gamma^j, y_\beta^j)$, the Hölder inequality and (4.30), we conclude that

$$\left|\left(\psi_\gamma^j, \widetilde{\psi}_{j,\beta}^{k,i}\right)\right| \le \int_{\mathcal{X}} \left|\psi_\gamma^j(x)\widetilde{\psi}_{j,\beta}^{k,i}(x)\right| d\mu(x) \lesssim e^{\nu\delta 2^{k-2}} \sqrt{\mu_\alpha^j} \int_{\mathcal{X}} \left|\psi_\gamma^j(x)\psi_\beta^j(x)\phi_\alpha^j(x)\right| d\mu(x)$$

$$\begin{aligned}
&\lesssim e^{\nu\delta 2^{k-2}} \int_{\mathcal{X}} \frac{e^{-\nu\delta^{-j} d(y_\gamma^j, x)}}{\sqrt{V(y_\gamma^j, \delta^j)}} \frac{e^{-\nu\delta^{-j} d(y_\beta^j, x)}}{\sqrt{V(y_\beta^j, \delta^j)}} e^{-\nu\delta^{-j} d(x_\alpha^j, x)}\, d\mu(x)\\
&\lesssim e^{\nu\delta 2^{k-2}} e^{-\frac{\nu}{2}\delta^{-j} d(y_\gamma^j, y_\beta^j)} e^{-\frac{\nu}{2}\delta^{-j} d(y_\beta^j, x_\alpha^j)}\\
&\quad\times \int_{\mathcal{X}} \frac{e^{-\frac{\nu}{2}\delta^{-j} d(y_\gamma^j, x)}}{\sqrt{V(y_\gamma^j, \delta^j)}} \frac{e^{-\frac{\nu}{2}\delta^{-j} d(y_\beta^j, x)}}{\sqrt{V(y_\beta^j, \delta^j)}}\, d\mu(x)\\
&\lesssim e^{-\nu\delta 2^{s-2}} \left\| \frac{e^{-\frac{\nu}{2}\delta^{-j} d(y_\gamma^j, \cdot)}}{\sqrt{V(y_\gamma^j, \delta^j)}} \right\|_{L^2(\mathcal{X})} \left\| \frac{e^{-\frac{\nu}{2}\delta^{-j} d(y_\beta^j, \cdot)}}{\sqrt{V(y_\beta^j, \delta^j)}} \right\|_{L^2(\mathcal{X})} \lesssim e^{-\nu\delta 2^{s-2}},
\end{aligned}$$

which, combined with Remark 4.26(ii) and Theorem 4.33, implies that

$$\mathrm{I}_2 \lesssim \sum_{s=0}^{\infty} 2^{(s+1)G_0} e^{-\nu\delta 2^{s-2}} \left\{ \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\mathscr{G}_j} \left|\left(h, \psi_\gamma^j\right)\right|^2 \right\}^{1/2} \lesssim \|h\|_{L^2(\mathcal{X})}.$$

Now we consider $\mathrm{I}_3$. We first estimate $|(\psi_\gamma^s, \widetilde{\psi}_{j,\beta}^{k,i})|$ for any $(j,\gamma)$, $(s,\gamma) \in \mathscr{C}$ with $d(y_\gamma^s, y_\beta^j) \ge \delta^{j+1}$ and $s \in \mathbb{Z} \cap (-\infty, j-1]$. By $\phi_\alpha^j \in V_j$, $\psi_\beta^j \in W_j$ and $V_j \perp W_j$ with $V_j$ and $W_j$ for all $j \in \mathbb{Z}$ as in Lemma 4.36, we have

$$\int_{\mathcal{X}} \widetilde{\psi}_{j,\beta}^{k,i}(x)\, d\mu(x) = e^{\nu\delta 2^{k-2}} \sqrt{\mu_\alpha^j} \int_{\mathcal{X}} \phi_\alpha^j(x) \psi_\beta^j(x)\, d\mu(x) = 0.$$

This, together with (4.11), (4.18), $\alpha \in \mathscr{A}_{j,\beta}^k$, $d(y_\gamma^s, y_\beta^j) \ge \delta^{j+1}$, the Hölder inequality, (4.30) and some arguments used in the estimate of [27, Proposition 3.4], further implies that

$$\begin{aligned}
\left|\left(\psi_\gamma^s, \widetilde{\psi}_{j,\beta}^{k,i}\right)\right| &= \left| \int_{\mathcal{X}} \left[\psi_\gamma^s(x) - \psi_\gamma^s\left(y_\beta^j\right)\right] \widetilde{\psi}_{j,\beta}^{k,i}(x)\, d\mu(x) \right|\\
&\lesssim e^{\nu\delta 2^{k-2}} \sqrt{\mu_\alpha^j} \int_{\mathcal{X}} \left|\psi_\gamma^s(x) - \psi_\gamma^s\left(y_\beta^j\right)\right| \left|\psi_\beta^j(x)\phi_\alpha^j(x)\right|\, d\mu(x)\\
&\lesssim e^{\nu\delta 2^{k-2}} \int_{\mathcal{X}} \left|\psi_\gamma^s(x) - \psi_\gamma^s\left(y_\beta^j\right)\right| \frac{e^{-\nu\delta^{-j} d(y_\beta^j, x)}}{\sqrt{V(y_\beta^j, \delta^j)}} e^{-\nu\delta^{-j} d(x_\alpha^j, x)}\, d\mu(x)\\
&\lesssim e^{\nu\delta 2^{k-2}} e^{-\frac{\nu}{2}\delta^{-j} d(y_\beta^j, x_\alpha^j)} \left\| \frac{e^{-\frac{\nu}{4}\delta^{-j} d(y_\beta^j, \cdot)}}{\sqrt{V(y_\beta^j, \delta^j)}} \right\|_{L^2(\mathcal{X})}\\
&\quad\times \left\{ \int_{\mathcal{X}} \left|\psi_\gamma^s(x) - \psi_\gamma^s\left(y_\beta^j\right)\right|^2 e^{-\frac{\nu}{2}\delta^{-j} d(y_\beta^j, x)}\, d\mu(x) \right\}^{1/2}\\
&\lesssim \left\{ e^{-\frac{\nu}{4}\delta^{-s} d(y_\gamma^s, y_\beta^j)} \sum_{t=0}^{j-s} \delta^{2t\eta} e^{-\frac{\nu}{2}\delta^{s+t+1-j}} \right\}^{1/2}.
\end{aligned}$$

By this and some arguments used in the estimate for $\mathrm{I}_3$ in the proof of [27, Proposition 3.4], we obtain

$$\mathrm{I}_3 \lesssim \|h\|_{L^2(\mathcal{X})}.$$

Finally, we deal with $\mathrm{I}_4$. To this end, we also need to estimate $|(\psi^s_\gamma, \widetilde{\psi}^{k,i}_{j,\beta})|$ for any $(j,\gamma)$, $(s,\gamma) \in \mathscr{C}$ with $d(y^s_\gamma, y^j_\beta) \geq \delta^{s+1}$ and $s \in \mathbb{Z} \cap [j+1,\infty)$. By $\psi^s_\gamma \in W_s$, $\phi^j_\alpha \in V_j \subset V_s$ and $W_s \perp V_s$ with $V_k$ and $W_k$ for any $k \in \mathbb{Z}$ as in Lemma 4.36, we have

$$\int_{\mathcal{X}} \psi^s_\gamma(x)\phi^j_\alpha(x)\,d\mu(x) = 0,$$

which, combined with (4.11), (4.18), the Hölder inequality, (4.30) and some arguments used in the proof of [27, Proposition 3.4], further implies that

$$\begin{aligned}
\left|\left(\psi^s_\gamma, \widetilde{\psi}^{k,i}_{j,\beta}\right)\right| &= \left|e^{\nu\delta 2^{k-2}}\sqrt{\mu^j_\alpha}\int_{\mathcal{X}} \psi^s_\gamma(x)\phi^j_\alpha(x)\psi^j_\beta(x)\,d\mu(x)\right|\\
&= \left|e^{\nu\delta 2^{k-2}}\sqrt{\mu^j_\alpha}\int_{\mathcal{X}} \psi^s_\gamma(x)\phi^j_\alpha(x)\left[\psi^j_\beta(x) - \psi^j_\beta\left(y^s_\gamma\right)\right]\,d\mu(x)\right|\\
&\lesssim e^{\nu\delta 2^{k-2}}\sqrt{\mu^j_\alpha}\int_{\mathcal{X}} \left|\psi^s_\gamma(x)\phi^j_\alpha(x)\right|\left|\psi^j_\beta(x) - \psi^j_\beta\left(y^s_\gamma\right)\right|\,d\mu(x)\\
&\lesssim e^{\nu\delta 2^{k-2}}\int_{\mathcal{X}} \left|\psi^j_\beta(x) - \psi^j_\beta\left(y^s_\gamma\right)\right|\frac{e^{-\nu\delta^{-s}d(y^s_\gamma,x)}}{\sqrt{V(y^s_\gamma,\delta^s)}}e^{-\nu\delta^{-j}d(x^j_\alpha,x)}\,d\mu(x)\\
&\lesssim e^{\nu\delta 2^{k-2}}\left\|\frac{e^{-\frac{\nu}{2}\delta^{-s}d(y^s_\gamma,\cdot)}}{\sqrt{V(y^s_\gamma,\delta^s)}}\right\|_{L^2(\mathcal{X})}\\
&\quad\times\left\{\int_{\mathcal{X}} \left|\psi^j_\beta(x) - \psi^j_\beta\left(y^s_\gamma\right)\right|^2 e^{-\nu\delta^{-s}d(y^j_\beta,x)}\,d\mu(x)\right\}^{1/2}\\
&\lesssim e^{-\frac{\nu}{4}\delta^{-j}d(y^s_\gamma,y^j_\beta)}\left\{\sum_{t=0}^{s-j}\delta^{2t\eta}e^{-\nu\delta^{j+t+1-s}}\right\}^{1/2}.
\end{aligned}$$

From this and the estimate of $\mathrm{I}_4$ in the proof of [27, Proposition 3.4], it follows that

$$\mathrm{I}_4 \lesssim \|h\|_{L^2(\mathcal{X})},$$

which, together with the estimates for $\mathrm{I}_1$, $\mathrm{I}_2$ and $\mathrm{I}_3$, then completes the proof of (4.29) and hence Proposition 4.38. $\square$

We also recall some estimates of integral kernels from [27, Proposition 3.5] as follows, where $s^j_\alpha$ is replaced by $\sqrt{\mu^j_\alpha}\phi^j_\alpha$. Let $k \in \mathbb{Z}$ and $i \in \{1,\ldots,m_k\}$ with $m_k$ as in (4.26). For any $(x,y) \in \{\mathcal{X}\times\mathcal{X}\}\setminus\{(x,x):\ x\in\mathcal{X}\}$, let

$$K_{k,i}(x,y) := \sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j}\widetilde{\psi}^{k,i}_{j,\beta}(x)\overline{\psi^j_\beta(y)}, \tag{4.31}$$

where $\mathscr{G}_j$ for any $j \in \mathbb{Z}$ is as in (4.13), and, for each $N \in \mathbb{N}$ and $x, y \in \mathcal{X}$, let

$$K^N_{k,i}(x,y) := \sum_{(j,\beta)\in\mathscr{C}_N} \widetilde{\psi}^{k,i}_{j,\beta}(x)\overline{\psi^j_\beta(y)}, \tag{4.32}$$

where $\mathscr{C}_N$ for any $N \in \mathbb{N}$ is as in (4.22).

Before proving Lemma 4.37, we give a new version of [27, Proposition 3.5].

**Proposition 4.39.** *Suppose that $(\mathcal{X}, d, \mu)$ is a metric measure space of homogeneous type, $N \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i \in \{1, \ldots, m_k\}$ with $m_k$ as in* (4.26)*. Let $K_{k,i}$, $K^N_{k,i}$ be defined as in* (4.31) *and* (4.32)*. Then*

$$K_{k,i},\, K^N_{k,i} \in L^1_{\rm loc}(\{\mathcal{X}\times\mathcal{X}\}\setminus\{(x,x):\ x\in\mathcal{X}\})$$

*and satisfy* (4.3)*,* (4.4) *and* (4.5) *with $s := \eta/2$ and $\eta$ as in* (4.12)*.*

*Proof.* Let $N \in \mathbb{N}$, $k \in \mathbb{Z}$, $i \in \{1, \ldots, m_k\}$, and the kernels $K_{k,i}$ and $K^N_{k,i}$ be defined as in (4.31) and (4.32), respectively. It suffices to show that $K_{k,i}$ satisfies (4.3), (4.4) and (4.5), since the proofs for $K^N_{k,i}$ are similar.

Now we show that $K_{k,i}$ satisfies (4.3). From (4.11), (4.18), $\alpha \in \mathscr{A}^k_{j,\beta}$ with $\mathscr{A}^k_{j,\beta}$ as in (4.25), and the estimate of H in the proof of [27, Proposition 3.5], we deduce that, for all $x, y \in \mathcal{X}$ with $x \neq y$,

$$\begin{aligned} |K_{k,i}(x,y)| &\lesssim e^{\nu\delta 2^{k-2}} \sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j} e^{-\nu\delta^{-j}d(x^j_\alpha,x)} \\ &\quad\times e^{-\frac{\nu}{2}\delta^{-j}d(y^j_\beta,x)} \left|\psi^j_\beta(x)\right|^{1/2} \frac{|\psi^j_\beta(y)|}{[V(y^j_\beta,\delta^j)]^{1/4}} \\ &\lesssim e^{\nu\delta 2^{k-2}} e^{-\frac{\nu}{2}\delta^{-j}d(y^j_\beta,x^j_\alpha)} \sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j} \left|\psi^j_\beta(x)\right|^{1/2} \frac{|\psi^j_\beta(y)|}{[V(y^j_\beta,\delta^j)]^{1/4}} \\ &\lesssim \sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j} \left|\psi^j_\beta(x)\right|^{1/2} \frac{|\psi^j_\beta(y)|}{[V(y^j_\beta,\delta^j)]^{1/4}} \lesssim \frac{1}{V(x,y)}, \end{aligned} \tag{4.33}$$

which, combined with (4.33), implies that $K_{k,i}$ satisfies (4.3).

We then show that $K_{k,i}$ satisfies (4.5). Let $x, y, \widetilde{y} \in \mathcal{X}$ with $0 < d(y,\widetilde{y}) \leq \frac{1}{2}d(x,y)$. From (4.11), (4.18) and $\alpha \in \mathscr{A}^k_{j,\beta}$ with $\mathscr{A}^k_{j,\beta}$ as in (4.25) and the estimate for J in the proof of [27, Proposition 3.5], together with $y^j_\beta := x^{j+1}_\beta$ for all $\beta \in \mathscr{G}_j$, we deduce that

$$\begin{aligned} &|K_{k,i}(x,y) - K_{k,i}(x,\widetilde{y})| \\ &\quad\lesssim e^{\nu\delta 2^{k-2}} \sum_{j\in\mathbb{Z}}\sum_{\beta\in\mathscr{G}_j} e^{-\nu\delta^{-j}d(x^j_\alpha,x)} e^{-\frac{\nu}{2}\delta^{-j}d(y^j_\beta,x)} \left|\psi^j_\beta(y) - \psi^j_\beta(\widetilde{y})\right| \\ &\quad\times \frac{|\psi^j_\beta(x)|^{1/2}}{[V(y^j_\beta,\delta^j)]^{1/4}} \end{aligned} \tag{4.34}$$

$$\lesssim e^{\nu\delta 2^{k-2}} e^{-\frac{\nu}{2}\delta 2^k} \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\psi_\beta^j(y) - \psi_\beta^j(\widetilde{y})\right| \frac{|\psi_\beta^j(x)|^{1/2}}{[V(y_\beta^j, \delta^j)]^{1/4}}$$

$$\lesssim \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\psi_\beta^j(y) - \psi_\beta^j(\widetilde{y})\right| \frac{|\psi_\beta^j(x)|^{1/2}}{[V(y_\beta^j, \delta^j)]^{1/4}} \lesssim \frac{1}{V(x,y)} \left[\frac{d(y,\widetilde{y})}{d(x,y)}\right]^\eta,$$

which further implies that $K$ satisfies (4.5).

Finally, we prove that $K$ satisfies (4.4). Let $x, \widetilde{x}, y \in \mathcal{X}$ with $0 < d(x,\widetilde{x}) \leq \frac{1}{2}d(x,y)$. From (4.11), (4.18) and $\alpha \in \mathscr{A}_{j,\beta}^k$, we deduce that

$$\begin{aligned}
&|K_{k,i}(x,y) - K_{k,i}(\widetilde{x},y)| \\
&\quad\leq e^{\nu\delta 2^{k-2}} \sqrt{\mu_\alpha^j} \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\phi_\alpha^j(x)\psi_\beta^j(x) - \phi_\alpha^j(\widetilde{x})\,\psi_\beta^j(\widetilde{x})\right| \left|\psi_\beta^j(y)\right| \\
&\quad\leq e^{\nu\delta 2^{k-2}} \sqrt{\mu_\alpha^j} \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left[\left|\phi_\alpha^j(x)\psi_\beta^j(x)\right|^{1/2} + \left|\phi_\alpha^j(\widetilde{x})\,\psi_\beta^j(\widetilde{x})\right|^{1/2}\right] \\
&\qquad\times \left|\phi_\alpha^j(x)\psi_\beta^j(x) - \phi_\alpha^j(\widetilde{x})\,\psi_\beta^j(\widetilde{x})\right|^{1/2} \left|\psi_\beta^j(y)\right| \\
&\quad\lesssim e^{\nu\delta 2^{k-2}} \left[\mu_\alpha^j\right]^{\frac14} \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left[e^{-\frac{\nu}{2}\delta^{-j} d(x_\alpha^j, x)} \frac{e^{-\frac{\nu}{2}\delta^{-j} d(y_\beta^j, x)}}{[V(y_\beta^j, \delta^j)]^{1/4}}\right. \\
&\qquad\left. + e^{-\frac{\nu}{2}\delta^{-j} d(x_\alpha^j, \widetilde{x})} \frac{e^{-\frac{\nu}{2}\delta^{-j} d(y_\beta^j, \widetilde{x})}}{[V(y_\beta^j, \delta^j)]^{1/4}}\right] \left|\phi_\alpha^j(x)\psi_\beta^j(x) - \phi_\alpha^j(\widetilde{x})\,\psi_\beta^j(\widetilde{x})\right|^{1/2} \left|\psi_\beta^j(y)\right| \\
&\quad\lesssim \left[\mu_\alpha^j\right]^{\frac14} \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\phi_\alpha^j(x)\psi_\beta^j(x) - \phi_\alpha^j(\widetilde{x})\,\psi_\beta^j(\widetilde{x})\right|^{1/2} \frac{|\psi_\beta^j(y)|}{[V(y_\beta^j, \delta^j)]^{1/4}} \\
&\quad\lesssim \left[\mu_\alpha^j\right]^{\frac14} \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\phi_\alpha^j(x)\right|^{1/2} \left|\psi_\beta^j(x) - \psi_\beta^j(\widetilde{x})\right|^{1/2} \frac{|\psi_\beta^j(y)|}{[V(y_\beta^j, \delta^j)]^{1/4}} \\
&\qquad + \left[\mu_\alpha^j\right]^{\frac14} \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\phi_\alpha^j(x) - \phi_\alpha^j(\widetilde{x})\right|^{1/2} \left|\psi_\beta^j(\widetilde{x})\right|^{1/2} \frac{|\psi_\beta^j(y)|}{[V(y_\beta^j, \delta^j)]^{1/4}} \quad =: \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}$$

By some arguments similar to those used in the estimates of (4.34), we have

$$\mathrm{J}_1 \lesssim \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left|\psi_\beta^j(x) - \psi_\beta^j(\widetilde{x})\right|^{1/2} \frac{|\psi_\beta^j(y)|}{[V(y_\beta^j, \delta^j)]^{1/4}} \lesssim \frac{1}{V(x,y)} \left[\frac{d(x,\widetilde{x})}{d(x,y)}\right]^\eta.$$

To estimate $\mathrm{J}_2$, by Theorem 4.31 and the estimate for B in the proof of [27, Proposition 3.5], we conclude that

$$\mathrm{J}_2 \lesssim \frac{1}{V(x,y)} \left[\frac{d(x,\widetilde{x})}{d(x,y)}\right]^{\eta/2}.$$

This, combined with the estimate for $\mathrm{J}_1$, implies that $K_{k,i}$ satisfies (4.4), which completes the proof of Proposition 4.39. □

The following notions of $(1,q,\eta)$-molecules are from [38] with a slight modification.

**Definition 4.40.** Let $q\in(1,\infty]$ and $\vec{\epsilon}:=\{\epsilon_k\}_{k\in\mathbb{N}}\subset[0,\infty)$ satisfying

$$\sum_{k=1}^{\infty} k\epsilon_k<\infty. \tag{4.35}$$

A function $M\in L^q(\mathcal{X})$ is called a $(1,q,\vec{\epsilon})$-*molecule* centered at a ball $B=B(x_0,r)$ with some $x_0\in\mathcal{X}$ and $r\in(0,\infty)$ if

(M1) $\|M\chi_B\|_{L^q(\mathcal{X})}\le[\mu(B)]^{1/q-1}$;
(M2) for all $k\in\mathbb{N}$, $\|M\chi_{B(x_0,2^kr)\setminus B(x_0,2^{k-1}r)}\|_{L^q(\mathcal{X})}\le\epsilon_k]\mu(2^kB)]^{1/q-1}$;
(M3) $\int_{\mathcal{X}}M(x)\,d\mu(x)=0$.

Then we state the molecular characterization of $H^1_{\rm at}(\mathcal{X})$ without resorting to the measure doubling condition (1.7); see [57, Theorem 3.2] for the details.

**Theorem 4.41** ([57])**.** *Assume that $(\mathcal{X},d,\mu)$ is a metric measure space of homogeneous type. Let $q\in(1,\infty]$ and $\vec{\epsilon}:=\{\epsilon_k\}_{k\in\mathbb{N}}\subset[0,\infty)$ satisfy* (4.35)*. Then there exists a positive constant $C$ such that, for any $(1,q,\vec{\epsilon})$-molecule $m$, it holds true that*

$$\|m\|_{H^1_{\rm at}(\mathcal{X})}\le C.$$

*Moreover, $f\in H^1_{\rm at}(\mathcal{X})$ if and only if there exist $(1,q,\vec{\epsilon})$-molecules $\{m_j\}_{j\in\mathbb{N}}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ such that*

$$f=\sum_{j=1}^{\infty}\lambda_j m_j$$

*holds true in $L^1(\mathcal{X})$. Furthermore, there exists a positive constant $C$, independent of $f$, such that*

$$C^{-1}\|f\|_{H^1_{\rm at}(\mathcal{X})}\le\inf\left\{\sum_{j=1}^{\infty}|\lambda_j|\right\}\le C\|f\|_{H^1_{\rm at}(\mathcal{X})},$$

*where the infimum is taken over all the decompositions of $f$ as above.*

**Remark 4.42.** The molecular characterization of $H^1_{\rm at}(\mathcal{X})$ appeared in [26, 27] has a problem that it only holds true on RD-spaces. For general spaces of homogeneous type, we need a revised version as in Theorem 4.41.

Now we state the following result from [27, Lemma 3.7] with some slight modifications.

**Lemma 4.43.** *Let $(\mathcal{X},d,\mu)$ be a metric measure space of homogeneous type. Then the bilinear operator $\Pi_1$ in* (4.24)*, originally defined for $f,\,g\in L^2(\mathcal{X})$ with finite wavelet decompositions as in* (4.23)*, can be extended to a bounded bilinear operator from $L^2(\mathcal{X})\times L^2(\mathcal{X})$ into $H^1_{\rm at}(\mathcal{X})$.*

*Proof.* Suppose that $f,\,g\in L^2(\mathcal{X})$ have finite wavelet decompositions as in (4.23), $\widetilde{s}^j_\alpha$ is as in Theorem 4.31 for all $(j,\alpha)\in\mathscr{A}$, with $\mathscr{A}$ as in (4.9), and, for all $(j,\alpha)\in\mathscr{A}$, $\mu^j_\alpha:=\mu(B(x^j_\alpha,\delta^j))$.

We first notice that, for each given $j \in \mathbb{Z} \cap [-M_2, M_2]$, with $M_2$ as in (4.23), $\alpha \in \mathscr{A}_j$ with $\mathscr{A}_j$ as in (4.6), and $\beta \in \mathscr{G}_j$, with $\mathscr{G}_j$ as in (4.13), by (4.13) and (4.8), $\beta \in \mathscr{G}_j$ if and only if $\beta \in \mathscr{G}_j$ and $d(x_\alpha^j, y_\beta^j) \geq \delta^{j+1}$. Furthermore, from the finite wavelet decomposition of $g$, we deduce that $Q_j g = 0$ for all $j \notin \mathbb{Z} \cap [-M_2, M_2]$. From these facts, (4.24) and Theorems 4.30 and 4.33, it follows that

$$
\begin{aligned}
(4.36) \qquad \Pi_1(f,g) &= \sum_{j=-M_2}^{M_2} (P_j f)(Q_j g) \\
&= \sum_{j=-M_2}^{M_2} \left[\sum_{\alpha \in \mathscr{A}_j} (f, \phi_\alpha^j)\, \phi_\alpha^j\right] \left[\sum_{\beta \in \mathscr{G}_j} \left(g, \psi_\beta^j\right) \psi_\beta^j\right] \\
&= \sum_{j=-M_2}^{M_2} \sum_{\alpha \in \mathscr{A}_j} \sum_{\{\beta \in \mathscr{G}_j:\ d(x_\alpha^j, y_\beta^j) \geq \delta^{j+1}\}} (f, \phi_\alpha^j) \left(g, \psi_\beta^j\right) \phi_\alpha^j \psi_\beta^j
\end{aligned}
$$

in $L^1(\mathcal{X})$. Now we show that

$$
\begin{aligned}
(4.37) \qquad T := \sum_{j=-M_2}^{M_2} \sum_{\beta \in \mathscr{G}_j} \sum_{\{\alpha \in \mathscr{A}_j:\ d(x_\alpha^j, y_\beta^j) \geq \delta^{j+1}\}} &\left|(f, \phi_\alpha^j)\right| \left|\left(g, \psi_\beta^j\right)\right| \\
&\times \int_{\mathcal{X}} \left|\phi_\alpha^j(x) \psi_\beta^j(x)\right| d\mu(x) < \infty.
\end{aligned}
$$

Indeed, by (4.11), (4.18), the Hölder inequality, (4.30) and (1.7), we conclude that

$$
\begin{aligned}
T &\lesssim \sum_{j=-M_2}^{M_2} \sum_{\beta \in \mathscr{G}_j} \sum_{\{\alpha \in \mathscr{A}_j:\ d(x_\alpha^j, y_\beta^j) \geq \delta^{j+1}\}} \left|(f, \phi_\alpha^j)\right| \left|\left(g, \psi_\beta^j\right)\right| \\
&\quad \times \int_{\mathcal{X}} \frac{e^{-\nu\delta^{-j} d(x, y_\beta^j)}}{\sqrt{V(y_\beta^j, \delta^j)}} \frac{e^{-\nu\delta^{-j} d(x, x_\alpha^j)}}{\sqrt{\mu_\alpha^j}}\, d\mu(x) \\
&\lesssim \sum_{j=-M_2}^{M_2} \sum_{\beta \in \mathscr{G}_j} \sum_{\{\alpha \in \mathscr{A}_j:\ d(x_\alpha^j, y_\beta^j) \geq \delta^{j+1}\}} \left|(f, \phi_\alpha^j)\right| \left|\left(g, \psi_\beta^j\right)\right| e^{-\frac{\nu}{2}\delta^{-j} d(x_\alpha^j, y_\beta^j)} \\
&\quad \times \int_{\mathcal{X}} \frac{e^{-\nu\delta^{-j} d(x, y_\beta^j)}}{\sqrt{V(y_\beta^j, \delta^j)}} \frac{e^{-\nu\delta^{-j} d(x, x_\alpha^j)}}{\sqrt{\mu_\alpha^j}}\, d\mu(x) \\
&\lesssim \sum_{j=-M_2}^{M_2} \sum_{\beta \in \mathscr{G}_j} \sum_{\{\alpha \in \mathscr{A}_j:\ d(x_\alpha^j, y_\beta^j) \geq \delta^{j+1}\}} \left|(f, \phi_\alpha^j)\right| \left|\left(g, \psi_\beta^j\right)\right| e^{-\frac{\nu}{2}\delta^{-j} d(x_\alpha^j, y_\beta^j)} \\
&\quad \times \left\{\int_{\mathcal{X}} \left[\frac{e^{-\frac{\nu}{2}\delta^{-j} d(x, y_\beta^j)}}{\sqrt{V(y_\beta^j, \delta^j)}}\right]^2 d\mu(x)\right\}^{1/2} \left\{\int_{\mathcal{X}} \left[\frac{e^{-\frac{\nu}{2}\delta^{-j} d(x, x_\alpha^j)}}{\sqrt{\mu_\alpha^j}}\right]^2 d\mu(x)\right\}^{1/2}
\end{aligned}
$$

$$\lesssim \sum_{j=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_j} \sum_{\{\alpha\in\mathscr{A}_j:\ d(x_\alpha^j, y_\beta^j)\ge\delta^{j+1}\}} \left|\left(f, \phi_\alpha^j\right)\right| \left|\left(g, \psi_\beta^j\right)\right| e^{-\frac{\nu}{2}\delta^{-j} d(x_\alpha^j, y_\beta^j)},$$

which, combined with the Hölder inequality, Lemma 4.29, Theorems 4.30 and 4.33, implies that

$$\begin{aligned}
T &\lesssim \left\{\sum_{j=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_j} \sum_{\{\alpha\in\mathscr{A}_j:\ d(x_\alpha^j, y_\beta^j)\ge\delta^{j+1}\}} \left|\left(f, \phi_\alpha^j\right)\right|^2 e^{-\nu\delta^{-j} d(x_\alpha^j, y_\beta^j)/2}\right\}^{1/2}\\
&\quad\times\left\{\sum_{j=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_j} \left|\left(g, \psi_\beta^j\right)\right|^2 \sum_{\alpha\in\mathscr{A}_j} e^{-\nu\delta^{-j} d(x_\alpha^j, y_\beta^j)/2}\right\}^{1/2}\\
&\lesssim \left\{\sum_{j=-M_2}^{M_2} \sum_{\alpha\in\mathscr{A}_j} \left|\left(f, \phi_\alpha^j\right)\right|^2 \sum_{\{\beta\in\mathscr{G}_j:\ d(x_\alpha^j, y_\beta^j)\ge\delta^{j+1}\}} e^{-\nu\delta^{-j} d(x_\alpha^j, y_\beta^j)/2}\right\}^{1/2}\\
&\quad\times\left\{\sum_{j=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_j} \left|\left(g, \psi_\beta^j\right)\right|^2\right\}^{1/2}\\
&\lesssim \left\{\sum_{j=-M_2}^{M_2} \sum_{\alpha\in\mathscr{A}_j} \left|\left(f, \phi_\alpha^j\right)\right|^2\right\}^{1/2} \|g\|_{L^2(\mathcal{X})} \lesssim M_2^{1/2}\|f\|_{L^2(\mathcal{X})}\|g\|_{L^2(\mathcal{X})} < \infty.
\end{aligned}$$

This shows that (4.37) holds true.

Recall that, for any $j \in \mathbb{N}\cap[-M_2, M_2]$, $\beta \in \mathscr{G}_j$, with $\mathscr{G}_j$ as in (4.13), and $k \in \mathbb{Z}_+$, $\mathscr{A}_{j,\beta}^k$ and $m_k$ are as in (4.25) and (4.26), respectively. Then, due to (4.36), (4.37) and the Fubini theorem, we write

$$\begin{aligned}
\Pi_1(f, g) &= \sum_{j=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_j} \sum_{\{\alpha\in\mathscr{A}_j:\ d(x_\alpha^j, y_\beta^j)\ge\delta^{j+1}\}} \left(f, \phi_\alpha^j\right)\left(g, \psi_\beta^j\right) \phi_\alpha^j \psi_\beta^j \\
&= \sum_{k=0}^{\infty} \sum_{j=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_j} \sum_{\alpha\in\mathscr{A}_{j,\beta}^k} \left(f, \phi_\alpha^j\right)\left(g, \psi_\beta^j\right) \phi_\alpha^j \psi_\beta^j \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} e^{-\nu\delta 2^{k-2}} \sum_{j=-M_2}^{M_2} \sum_{\beta\in\mathscr{G}_j} \left(f, \frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right) \\
&\quad\times \left(g, \psi_\beta^j\right) e^{\nu\delta 2^{k-2}} \sqrt{\mu^j_{\alpha^i_{j,\beta}}}\, \phi^j_{\alpha^i_{j,\beta}} \psi_\beta^j
\end{aligned} \tag{4.38}$$

in $L^1(\mathcal{X})$.

In order to estimate $\Pi_1(f,g)$, we need to recall the following operator $U_{k,\,i}$ from [27] for any $k\in\mathbb{Z}_+$ and $i\in\{1,\ldots,m_k\}$. For any $(j,\beta)\in\mathscr{C}$, let

$$U_{k,\,i}\left(\psi_\beta^j\right):=\widetilde{\psi}_{j,\,\beta}^{k,\,i}, \tag{4.39}$$

where $\widetilde{\psi}_{j,\,\beta}^{k,\,i}$ is as in (4.27) and $\psi_\beta^j$ as in Theorem 4.33 with $(k,\alpha)$ replaced by $(j,\beta)$. We now recall from [27, Section 3] that $U_{k,\,i}$ can be extended to a bounded linear operator on $L^2(\mathcal{X})$ and on $H^1_{\rm at}(\mathcal{X})$.

We claim that, for each $k\in\mathbb{Z}_+$ and $i\in\{1,\ldots,m_k\}$,

$$\sum_{j=-M_2}^{M_2}\sum_{\beta\in\mathscr{G}_j}\left(f,\frac{\phi^j_{\alpha^i_{j,\,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\,\beta}}}}\right)\left(g,\psi_\beta^j\right)\psi_\beta^j\in H^1_{\rm at}(\mathcal{X}). \tag{4.40}$$

Indeed, from $\alpha^i_{j,\,\beta}\in\mathscr{A}^k_{j,\,\beta}$ with $\mathscr{A}^k_{j,\,\beta}$ as in (4.25) and (1.7), we deduce that

$$V\left(y^j_\beta,\delta^j\right)\le V\left(x^j_{\alpha^i_{j,\,\beta}},2^{k+2}\delta^j\right)\lesssim 2^{nk}V\left(x^j_{\alpha^i_{j,\,\beta}},\delta^j\right)\sim 2^{nk}\mu^j_{\alpha^i_{j,\,\beta}}. \tag{4.41}$$

Moreover, from the proof of [26, Lemma 3.7], it follows that, for any $j\in\mathbb{Z}$ and $\beta\in\mathscr{G}_j$, $\frac{\psi^j_\beta}{\sqrt{V(y^j_\beta,\delta^j)}}$ is a $(1,2,\eta)$-molecule multiplied by a positive constant independent of $j$ and $\beta$. Thus, by this, the completion of $H^1_{\rm at}(\mathcal{X})$, Theorem 4.41, (4.41), the Hölder inequality, Theorems 4.30 and 4.33, the fact that, for any $\beta\in\mathscr{G}_j$, there exist at most $m_k$ points $\alpha^i_{j,\,\beta}$ in $\mathscr{A}^k_{j,\,\beta}\subset\mathscr{A}_j$ corresponding to $\beta$, $V(y^j_\beta,\delta^j)\subset V(x_{\alpha^i_{j,\,\beta}},2^{k+2}\delta^j)$ and (1.7), we conclude that

$$\begin{aligned}
&\left\|\sum_{j=-M_2}^{M_2}\sum_{\beta\in\mathscr{G}_j}\left(f,\frac{\phi^j_{\alpha^i_{j,\,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\,\beta}}}}\right)\left(g,\psi_\beta^j\right)\psi_\beta^j\right\|_{H^1_{\rm at}(\mathcal{X})}\\
&\quad\lesssim\sum_{j=-M_2}^{M_2}\sum_{\beta\in\mathscr{G}_j}\left|\left(f,\phi^j_{\alpha^i_{j,\,\beta}}\right)\right|\left|\left(g,\psi^j_\beta\right)\right|\sqrt{\frac{V(y^j_\beta,\delta^j)}{\mu^j_{\alpha^i_{j,\,\beta}}}}\\
&\quad\lesssim 2^{nk/2}\sum_{j=-M_2}^{M_2}\left[\sum_{\beta\in\mathscr{G}_j}\left|\left(f,\phi^j_{\alpha^i_{j,\,\beta}}\right)\right|^2\right]^{1/2}\left[\sum_{\beta\in\mathscr{G}_j}\left|\left(g,\psi^j_\beta\right)\right|^2\right]^{1/2}\\
&\quad\lesssim 2^{nk/2}m_k^{1/2}\sum_{j=-M_2}^{M_2}\left[\sum_{\alpha\in\mathscr{A}_j}\left|\left(f,\phi^j_\alpha\right)\right|^2\right]^{1/2}\|g\|_{L^2(\mathcal{X})}\\
&\quad\lesssim 2^{nk/2}m_k^{1/2}M_2\|f\|_{L^2(\mathcal{X})}\|g\|_{L^2(\mathcal{X})}<\infty.
\end{aligned}$$

This finishes the proof of the above claim (4.40).

By (4.38), (4.39), the above claim and the boundedness of $U_{k,i}$ on $H^1_{\rm at}(\mathcal{X})$ uniformly with respect to $k$ and $i$, we conclude that

$$\Pi_1(f,g)=\sum_{k=0}^{\infty}\sum_{i=1}^{m_k}e^{-\nu\delta 2^{k-2}}\sum_{j=-M_2}^{M_2}\sum_{\beta\in\mathscr{G}_j}\left(f,\frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right)\left(g,\psi^j_\beta\right)U_{k,i}\left(\psi^j_\beta\right)$$
$$=\sum_{k=0}^{\infty}\sum_{i=1}^{m_k}e^{-\nu\delta 2^{k-2}}U_{k,i}\left(\sum_{j=-M_2}^{M_2}\sum_{\beta\in\mathscr{G}_j}\left(f,\frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right)\left(g,\psi^j_\beta\right)\psi^j_\beta\right)$$

in $L^1(\mathcal{X})$. From the above claim, (4.40), together with the boundedness of $U_{k,i}$ on $H^1_{\rm at}(\mathcal{X})$ uniformly with respect to $k$ and $i$, and [26, Theorem 4.4], we deduce that

(4.42)
$$\mathrm{L}:=\sum_{k=0}^{\infty}\sum_{i=1}^{m_k}e^{-\nu\delta 2^{k-2}}\left\|U_{k,i}\left(\sum_{j=-M_2}^{M_2}\sum_{\beta\in\mathscr{G}_j}\left(f,\frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right)\left(g,\psi^j_\beta\right)\psi^j_\beta\right)\right\|_{H^1_{\rm at}(\mathcal{X})}$$
$$\lesssim\sum_{k=0}^{\infty}\sum_{i=1}^{m_k}e^{-\nu\delta 2^{k-2}}\left\|\sum_{j=-M_2}^{M_2}\sum_{\beta\in\mathscr{G}_j}\left(f,\frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right)\left(g,\psi^j_\beta\right)\psi^j_\beta\right\|_{H^1_{\rm at}(\mathcal{X})}$$
$$\lesssim\sum_{k=0}^{\infty}\sum_{i=1}^{m_k}e^{-\nu\delta 2^{k-2}}\left\|\left\{\sum_{(j,\gamma,\beta)\in\mathscr{I}}\left|\left(f,\frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right)\left(g,\psi^j_{\gamma,\beta}\right)\right|^2\frac{\chi_{Q^j_\gamma}}{\mu(Q^j_\gamma)}\right\}^{1/2}\right\|_{L^1(\mathcal{X})},$$

where $\mathscr{I}$ is as in (4.20).

Moreover, by $\alpha^i_{j,\beta}\in\mathscr{A}^k_{j,\beta}$, $(j+1,\beta)\le(j,\gamma)$ and Remark 4.28(i), we obtain

$$d\left(x^j_{\alpha^i_{j,\beta}},x^j_\gamma\right)\le d\left(x^j_{\alpha^i_{j,\beta}},y^j_\beta\right)+d\left(y^j_\beta,x^j_\gamma\right)<2^{k+1}\delta^{j+1}+2\delta^{j+1}\le 2^{k+2}\delta^{j+1},$$

which, combined with Theorem 4.27(iv), implies that

$$Q^j_\gamma\subset B(x^j_\gamma,4\delta^j)\subset B(x^j_{\alpha^i_{j,\beta}},2^{k+3}\delta^j).$$

From these inclusion relations, (4.11) and (1.7), we further deduce that, for all $x\in\mathcal{X}$,

$$\left|\left(f,\frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right)\right|\frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)}$$

$$
\begin{aligned}
&\le \left|\left(f, \frac{\phi^j_{\alpha^i_{j,\beta}}}{\sqrt{\mu^j_{\alpha^i_{j,\beta}}}}\right)\right| \chi_{B(x^j_{\alpha^i_{j,\beta}}, 2^{k+3}\delta^j)}(x) \frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)}\\
&\lesssim \frac{[\widetilde{C}_{(\mathcal{X})}]^k}{V(x^j_{\alpha^i_{j,\beta}}, 2^{k+3}\delta^j)} \int_{\mathcal{X}} |f(y)| e^{-\nu\delta^{-j} d(y, x^j_{\alpha^i_{j,\beta}})}\, d\mu(y) \chi_{B(x^j_{\alpha^i_{j,\beta}}, 2^{k+3}\delta^j)}(x) \frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)}\\
&\lesssim \frac{[\widetilde{C}_{(\mathcal{X})}]^k}{V(x^j_{\alpha^i_{j,\beta}}, 2^{k+3}\delta^j)} \int_{B(x^j_{\alpha^i_{j,\beta}}, 2^{k+3}\delta^j)} |f(y)|\, d\mu(y) \chi_{B(x^j_{\alpha^i_{j,\beta}}, 2^{k+3}\delta^j)}(x) \frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)}\\
&\quad + \frac{[\widetilde{C}_{(\mathcal{X})}]^k}{V(x^j_{\alpha^i_{j,\beta}}, 2^{k+3}\delta^j)} \sum_{s=1}^\infty e^{-\nu 2^{s+k+2}} \int_{B(x^j_{\alpha^i_{j,\beta}}, 2^{s+k+3}\delta^j)\setminus B(x^j_{\alpha^i_{j,\beta}}, 2^{s+k+2}\delta^j)} |f(y)|\, d\mu(y)\\
&\quad \times \chi_{B(x^j_{\alpha^i_{j,\beta}}, 2^{s+k+3}\delta^j)}(x) \frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)}\\
&\lesssim [\widetilde{C}_{(\mathcal{X})}]^k M(f)(x) \frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)} + [\widetilde{C}_{(\mathcal{X})}]^k \sum_{s=1}^\infty 2^{sn} e^{-\nu 2^{s+k+2}} M(f)(x) \frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)}\\
&\lesssim [\widetilde{C}_{(\mathcal{X})}]^k M(f)(x) \frac{\chi_{Q^j_\gamma}(x)}{\mu(Q^j_\gamma)},
\end{aligned}
$$

which, together with (4.42), $m_k := N_0 2^{(k+1)G_0}$, the Hölder inequality, the boundedness of the Hardy-Littlewood maximal function $M$ on $L^2(\mathcal{X})$ and Theorem 4.33, further implies that

$$
\begin{aligned}
\mathrm{L} &\lesssim \sum_{k=0}^\infty \left[\widetilde{C}_{(\mathcal{X})}\right]^k m_k e^{-\nu\delta 2^{k-2}} \left\| M(f) \left\{ \sum_{(j,\gamma,\beta)\in\mathscr{I}} \left|\left(g, \psi^j_{\gamma,\beta}\right)\right|^2 \frac{\chi_{Q^j_\gamma}}{\mu(Q^j_\gamma)} \right\}^{1/2} \right\|_{L^1(\mathcal{X})}\\
&\lesssim \sum_{k=0}^\infty \left[\widetilde{C}_{(\mathcal{X})}\right]^k m_k e^{-\nu\delta 2^{k-2}} \|M(f)\|_{L^2(\mathcal{X})} \left\{ \sum_{(j,\gamma,\beta)\in\mathscr{I}} \left|\left(g, \psi^j_{\gamma,\beta}\right)\right|^2 \right\}^{1/2}\\
&\lesssim \sum_{k=0}^\infty \left[\widetilde{C}_{(\mathcal{X})}\right]^k m_k e^{-\nu\delta 2^{k-2}} \|f\|_{L^2(\mathcal{X})} \|g\|_{L^2(\mathcal{X})} \lesssim \|f\|_{L^2(\mathcal{X})} \|g\|_{L^2(\mathcal{X})}.
\end{aligned}
$$

This, combined with the completion of $H^1_{\mathrm{at}}(\mathcal{X})$, then implies that $\Pi_1(f,g) \in H^1_{\mathrm{at}}(\mathcal{X})$ and

$$
\|\Pi_1(f,g)\|_{H^1_{\mathrm{at}}(\mathcal{X})} \lesssim \mathrm{L} \lesssim \|f\|_{L^2(\mathcal{X})} \|g\|_{L^2(\mathcal{X})},
$$

which, together with the fact that the functions in $L^2(\mathcal{X})$ with finite wavelet decompositions as in (4.23) are dense in $L^2(\mathcal{X})$ and a standard density argument, further completes the proof of Lemma 4.43. □

By [2, Corollary 11.2] (see also [27, (4.1)]), we know that, for any $j \in \mathbb{Z}$, $\beta \in \mathscr{G}_j$ and $g \in \mathrm{BMO}(\mathcal{X})$, $\langle g, \psi^j_\beta \rangle$ is well defined and there exists a positive constant $C$

such that, for all $g \in \mathrm{BMO}\,(\mathcal{X})$,

$$\left|\left\langle g, \psi_\beta^j \right\rangle\right| \le C\|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sqrt{V(y_\beta^j, \delta^j)}. \tag{4.43}$$

It was shown in [27, Section 4] that $Q_j g := \sum_{\beta\in\mathscr{G}_j} \left\langle g, \psi_\beta^j \right\rangle \psi_\beta^j$ is also pointwisely well defined.

Analogously, let $j \in \mathbb{Z}$ and $g \in \mathrm{BMO}\,(\mathcal{X})$.

$$P_j g := \sum_{\alpha\in\mathscr{A}_j} \left\langle g, \phi_\alpha^j \right\rangle \phi_\alpha^j$$

is pointwisely well defined.

Now we are ready to prove Lemma 4.37.

*Proof of Lemma 4.37.* We first show that, for each $(1,2)$-atom $a$ related to a ball $B_0 := B(x_0, r_0)$, with some $x_0 \in \mathcal{X}$ and $r_0 \in (0,\infty)$, and $g \in \mathrm{BMO}\,(\mathcal{X})$, $\Pi_1(a,g) \in H^1_{\mathrm{at}}(\mathcal{X})$ and

$$\|\Pi_1(a,g)\|_{H^1_{\mathrm{at}}(\mathcal{X})} \lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}, \tag{4.44}$$

where the implicit positive constant is independent of $a$ and $g$.

Indeed, let $k_0 \in \mathbb{Z}$ satisfy $\delta^{k_0+1} \le r_0 < \delta^{k_0}$ and $C_4$ be a sufficiently large positive constant which is determined later. We formally write

$$\begin{aligned}
\Pi_1(a,g) &= \sum_{j=k_0+1}^{\infty} \left[\sum_{\alpha\in\mathscr{A}_j} \left(a, \phi_\alpha^j\right) \phi_\alpha^j\right] \left[\sum_{\{\beta\in\mathscr{G}_j:\ y_\beta^j\in C_4B_0\}} \left\langle g, \psi_\beta^j \right\rangle \psi_\beta^j\right] \\
&\quad + \sum_{j=k_0+1}^{\infty} \left[\sum_{\alpha\in\mathscr{A}_j} \left(a, \phi_\alpha^j\right) \phi_\alpha^j\right] \left[\sum_{\{\beta\in\mathscr{G}_j:\ y_\beta^j\notin C_4B_0\}} \left\langle g, \psi_\beta^j \right\rangle \psi_\beta^j\right] \\
&\quad + \sum_{j=-\infty}^{k_0} \left[\sum_{\alpha\in\mathscr{A}_j} \left(a, \phi_\alpha^j\right) \phi_\alpha^j\right] \left[\sum_{\beta\in\mathscr{G}_j} \left\langle g, \psi_\beta^j \right\rangle \psi_\beta^j\right] \\
&=: \Pi_1^{(1)}(a,g) + \Pi_1^{(2)}(a,g) + \Pi_1^{(3)}(a,g).
\end{aligned}$$

Let

$$g_1 := \sum_{\{\ell\in\mathbb{Z}:\ \delta^\ell\le r_0\}} \sum_{\{\theta\in\mathscr{G}_\ell:\ y_\theta^\ell\in C_4B_0\}} \left\langle g, \psi_\theta^\ell \right\rangle \psi_\theta^\ell.$$

From the proof of [27, Theorem 4.9], it follows that $g^{(1)} \in L^2(\mathcal{X})$ and

$$\left\|g^{(1)}\right\|_{L^2(\mathcal{X})} \lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sqrt{\mu(B_0)}.$$

By this and $a \in L^2(\mathcal{X})$, combined with Lemma 4.43, we conclude that $\Pi_1^{(1)}(a,g) = \Pi_1(a, g_1)$ belongs to $H^1_{\mathrm{at}}(\mathcal{X})$, which, together with Lemma 4.43 and an argument similar to that used in the estimate for $\Pi_3^{(1)}(a,g)$ in the proof of [27, Theorem 4.9], implies that

$$\left\|\Pi_1^{(1)}(a,g)\right\|_{H^1_{\rm at}(\mathcal{X})} = \|\Pi_1(a,g_1)\|_{H^1_{\rm at}(\mathcal{X})} \lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

Then we estimate $\Pi_1^{(2)}(a,g)$. By [57, Theorem 3.2], we know that, for all $j\in\mathbb{Z}$, $\alpha\in\mathscr{A}_j$, and $\beta\in\mathscr{G}_j$ with $\mathscr{G}_j$ as in (4.13), there exists $\vec{\epsilon}:=\{\epsilon_k\}_{k\in\mathbb{N}}$ satisfying (4.35) such that

$$a^j_{\alpha,\beta} := e^{\frac{\nu}{2}\delta^{-j}d(x^j_\alpha,\,y^j_\beta)}\phi^j_\alpha\psi^j_\beta \quad \text{is a } (1,2,\vec{\epsilon})\text{−molecule,} \tag{4.45}$$

related to the ball $B(x^j_\alpha,\delta^j)$, multiplied by a positive harmless constant independent of $k$, $\alpha$ and $\beta$.

By Theorem 4.31 and $r_0<\delta^{k_0}<\delta^j$ for any $j>k_0$, we have

$$\begin{aligned}\left|\left(a,\phi^j_\alpha\right)\right| &\lesssim \int_{B_0}|a(x)|\frac{1}{\sqrt{\mu^j_\alpha}}e^{-\nu\delta^{-j}d(x_0,x^j_\alpha)}\,d\mu(x)\\ &\lesssim \int_{B_0}|a(x)|\frac{1}{\sqrt{\mu^j_\alpha}}e^{-\nu\delta^{-j}d(x_0,x^j_\alpha)}e^{\nu\delta^{-j}d(x_0,x)}\,d\mu(x)\\ &\lesssim e^{\nu\delta^{-j}r_0}\int_{B_0}|a(x)|\frac{1}{\sqrt{\mu^j_\alpha}}e^{-\nu\delta^{-j}d(x_0,x^j_\alpha)}\,d\mu(x)\\ &\lesssim \frac{1}{\sqrt{\mu^j_\alpha}}e^{-\nu\delta^{-j}d(x_0,x^j_\alpha)}\|a\|_{L^1(\mathcal{X})} \lesssim \frac{1}{\sqrt{\mu^j_\alpha}}e^{-\nu\delta^{-j}d(x_0,x^j_\alpha)}.\end{aligned} \tag{4.46}$$

From (4.45), (4.43), (4.11), (1.7) and Lemma 4.29, we deduce that

$$\begin{aligned}\mathrm{A} &:= \sum_{j=k_0+1}^{\infty}\sum_{\alpha\in\mathscr{A}_j}\sum_{\{\beta\in\mathscr{G}_j:\ y^j_\beta\notin C_4B_0\}}\left|\left(a,\phi^j_\alpha\right)\right|\left|\left\langle g,\psi^j_\beta\right\rangle\right|\left\|\phi^j_\alpha\psi^j_\beta\right\|_{H^1_{\rm at}(\mathcal{X})}\\ &\lesssim \sum_{j=k_0+1}^{\infty}\sum_{\alpha\in\mathscr{A}_j}\sum_{\{\beta\in\mathscr{G}_j:\ y^j_\beta\notin C_4B_0\}}\left|\left(a,\phi^j_\alpha\right)\right|\left|\left\langle g,\psi^j_\beta\right\rangle\right|e^{-\frac{\nu}{2}\delta^{-j}d(x^j_\alpha,\,y^j_\beta)}\\ &\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\|a\|_{L^1(\mathcal{X})}\sum_{j=k_0+1}^{\infty}\sum_{\alpha\in\mathscr{A}_j}e^{-\nu\delta^{-j}d(x_0,x^j_\alpha)}\\ &\quad\times\sum_{\{\beta\in\mathscr{G}_j:\ y^j_\beta\notin C_4B_0\}}\left[\frac{V(y^j_\beta,\delta^j)}{V(x^j_\alpha,\delta^j)}\right]^{1/2}e^{-\frac{\nu}{2}\delta^{-j}d(y^j_\beta,\,x^j_\alpha)}\\ &\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sum_{j=k_0+1}^{\infty}\sum_{\alpha\in\mathscr{A}_j}e^{-\nu\delta^{-j}d(x_0,x^j_\alpha)}\sum_{\{\beta\in\mathscr{G}_j:\ y^j_\beta\notin C_4B_0\}}e^{-\frac{\nu}{4}\delta^{-j}d(y^j_\beta,\,x^j_\alpha)}\\ &\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sum_{j=k_0+1}^{\infty}\sum_{\alpha\in\mathscr{A}_j}e^{-\frac{3\nu}{4}\delta^{-j}d(x_0,x^j_\alpha)}\sum_{\{\beta\in\mathscr{G}_j:\ y^j_\beta\notin C_4B_0\}}e^{-\frac{\nu}{4}\delta^{-j}d(y^j_\beta,\,x_0)}\end{aligned}$$

$$\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})} \sum_{j=k_0+1}^{\infty} \sum_{\{\beta\in\mathscr{G}_j:\ y_\beta^j\notin C_4B_0\}} e^{-\frac{\nu}{4}\delta^{-j}d(y_\beta^j,x_0)}.$$

This, combined with $\delta^{k_0+1} \le r_0 < \delta^{k_0}$, implies that

$$\begin{aligned}
\mathrm{A} &\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})} \sum_{j=k_0+1}^{\infty} \sum_{\{\beta\in\mathscr{G}_j:\ y_\beta^j\notin C_4B_0\}} e^{-\frac{\nu}{4}\delta^{-j}d(y_\beta^j,x_0)}\\
&\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})} \sum_{j=k_0+1}^{\infty}\sum_{t=0}^{\infty} \sum_{\{\beta\in\mathscr{G}_j:\ y_\beta^j\in 2^{t+1}C_4B_0\setminus 2^tC_4B_0\}} e^{-\frac{\nu}{4}C_4\delta^{k_0-j+1}2^t}\\
&\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})} \sum_{j=k_0+1}^{\infty}\sum_{t=0}^{\infty} 2^{-tM_0}\delta^{(j-k_0)M_0}\delta^{(k_0-j)G_0}2^{tG_0} \lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})},
\end{aligned}$$

where $M_0$ and $C_4$ are chosen to be sufficiently large positive constants such that $M_0 > 2G_0$, with $G_0$ as in Remark 4.26(ii). Therefore, from the completion of $H^1_{\mathrm{at}}(\mathcal{X})$, we deduce that $\Pi_1^{(2)}(a,g) \in H^1_{\mathrm{at}}(\mathcal{X})$ and

$$\left\|\Pi_1^{(2)}(a,g)\right\|_{H^1_{\mathrm{at}}(\mathcal{X})} \le \mathrm{A} \lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

Finally, we deal with $\Pi_1^{(3)}(a,g)$. We first estimate $|(a,\phi_\alpha^j)|$ for all $j\in\mathbb{Z}\cap(-\infty,k_0]$ and $\alpha\in\mathscr{A}_j$ with $x_\alpha^j\in B(x_0,9\delta^j)$. By $\int_{\mathcal{X}} a(x)\,d\mu(x)=0$, $r_0<\delta^{k_0}\le\delta^j$ for all $j\le k_0$, and (4.12), we have

$$\begin{aligned}
(4.47)\qquad \left|(a,\phi_\alpha^j)\right| &\le \int_{B_0} |a(x)|\left|\phi_\alpha^j(x)-\phi_\alpha^j(x_0)\right| d\mu(x)\\
&\lesssim \int_{B_0} |a(x)| \left[\frac{d(x,x_0)}{\delta^j}\right]^\eta \frac{1}{\sqrt{\mu_\alpha^j}} e^{-\nu\delta^{-j}d(x_0,x_\alpha^j)}\,d\mu(x)\\
&\lesssim \frac{1}{\sqrt{\mu_\alpha^j}}\delta^{(k_0-j)\eta}e^{-\nu\delta^{-j}d(x_0,x_\alpha^j)}\|a\|_{L^1(\mathcal{X})}\\
&\lesssim \frac{1}{\sqrt{\mu_\alpha^j}}\delta^{(k_0-j)\eta}e^{-\nu\delta^{-j}d(x_0,x_\alpha^j)}.
\end{aligned}$$

From (4.45), (4.47), (1.7), (4.43) and Lemma 4.29, it follows that

$$\begin{aligned}
&\sum_{j=-\infty}^{k_0}\sum_{\alpha\in\mathscr{A}_j}\sum_{\beta\in\mathscr{G}_j}\left|(a,\phi_\alpha^j)\right|\left|\left\langle g,\psi_\beta^j\right\rangle\right|\left\|\phi_\alpha^j\psi_\beta^j\right\|_{H^1_{\mathrm{at}}(\mathcal{X})}\\
&\quad\lesssim \sum_{j=-\infty}^{k_0}\sum_{\alpha\in\mathscr{A}_j}\sum_{\beta\in\mathscr{G}_j}\left|(a,\phi_\alpha^j)\right|\left|\left\langle g,\psi_\beta^j\right\rangle\right| e^{-\frac{\nu}{2}\delta^{-j}d(y_\beta^j,y_\gamma^j)}\\
&\quad\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sum_{j=-\infty}^{k_0}\delta^{(k_0-j)\eta}\sum_{\alpha\in\mathscr{A}_j} e^{-\nu\delta^{-j}d(x_0,x_\alpha^j)}
\end{aligned}$$

$$\times \sum_{\{\beta\in\mathscr{G}_j:\ d(x_\alpha^j,y_\beta^j)\ge\delta^{j+1}\}} e^{-\frac{\nu}{2}\delta^{-j}d(x_\alpha^j,y_\beta^j)}\left[\frac{V(y_\beta^j,\delta^j)}{V(x_\alpha^j,\delta^j)}\right]^{1/2}$$

$$\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sum_{j=-\infty}^{k_0}\delta^{(k_0-j)\eta}\sum_{\alpha\in\mathscr{A}_j} e^{-\nu\delta^{-j}d(x_0,x_\alpha^j)}$$

$$\times \sum_{\{\beta\in\mathscr{G}_j:\ d(x_\alpha^j,y_\beta^j)\ge\delta^{j+1}\}} e^{-\frac{\nu}{2}\delta^{-j}d(x_\alpha^j,y_\beta^j)}\left[\frac{d(y_\beta^j,x_\alpha^j)+\delta^j}{\delta^j}\right]^{1/2}$$

$$\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sum_{j=-\infty}^{k_0}\delta^{(k_0-j)\eta}\sum_{\{\beta\in\mathscr{G}_j:\ d(x_\alpha^j,y_\beta^j)\ge\delta^{j+1}\}} e^{-\frac{\nu}{4}\delta^{-j}d(x_\alpha^j,y_\beta^j)}$$

$$\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}\sum_{j=-\infty}^{k_0}\delta^{(k_0-j)\eta}\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})},$$

which, together with the completion of $H^1_{\mathrm{at}}(\mathcal{X})$, implies that $\Pi_1^{(3)}(a,g)\in H^1_{\mathrm{at}}(\mathcal{X})$ and

$$\left\|\Pi_1^{(3)}(a,g)\right\|_{H^1_{\mathrm{at}}(\mathcal{X})}\lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

From this and the estimates of $\Pi_1^{(1)}(a,g)$ and $\Pi_1^{(2)}(a,g)$, we deduce that $\Pi_1(a,g)$ belongs to $H^1_{\mathrm{at}}(\mathcal{X})$ and (4.44) holds true, which, combined with [27, Theorem 4.7] and an argument similar to that used in the proof of [27, Theorem 4.9], further implies that $\Pi_1$ can be extended to a bounded bilinear operator from $H^1_{\mathrm{at}}(\mathcal{X})\times \mathrm{BMO}\,(\mathcal{X})$ into $H^1_{\mathrm{at}}(\mathcal{X})$. This finishes the proof of Lemma 4.37. □

Now we give the following revised version of [27, Theorem 4.16].

**Lemma 4.44.** *Let $\epsilon\in(0,1]$, $\varrho,\vartheta\in(0,\epsilon]$ and $(\mathcal{X},d,\mu)$ be a metric measure space of homogeneous type. Then, for any $(1,2)$-atom $a$ related to a ball $B_0$ and $g\in \mathrm{BMO}\,(\mathcal{X})$, the bilinear operator $\Pi_2$, defined by setting*

$$\begin{aligned}\Pi_2(a,g) &:= ag_{B_0}+\widetilde{\Pi}_2(a,g)\\ &:= ag_{B_0}+\sum_{j\in\mathbb{Z}}\sum_{\alpha\in\mathscr{A}_j}\sum_{\beta\in\mathscr{G}_j}\left(g-g_{B_0},\phi_\alpha^j\right)\left(a,\psi_\beta^j\right)\phi_\alpha^j\psi_\beta^j \quad\text{in}\quad (\mathcal{G}_0^\epsilon(\varrho,\vartheta))',\end{aligned}$$

*can be extended to a bounded bilinear operator from $H^1_{\mathrm{at}}(\mathcal{X})\times\mathrm{BMO}^+(\mathcal{X})$ into $H^{\log}(\mathcal{X})$. Furthermore, it holds true that*

$$\Pi_2(a,g)=h+am_{B_0}(g),$$

*where $h\in H^1_{\mathrm{at}}(\mathcal{X})$ satisfies that there exists a positive constant $C$, independent of $g$ and $h$, such that $\|h\|_{H^1_{\mathrm{at}}(\mathcal{X})}\le C\|g\|_{\mathrm{BMO}\,(\mathcal{X})}$.*

*Proof.* We first show that, for any $(1,2)$-atom $a$ supported in a ball $B_0:=B(x_0,r_0)$, with some $x_0\in\mathcal{X}$ and $r_0\in(0,\infty)$, and $g\in\mathrm{BMO}\,(\mathcal{X})$, $\Pi_2(a,g)\in H^{\log}(\mathcal{X})$ and

$$\|\Pi_2(a,g)\|_{H^{\log}(\mathcal{X})}\lesssim\|g\|_{\mathrm{BMO}^+(\mathcal{X})}, \tag{4.48}$$

where the implicit positive constant is independent of $a$ and $g$.

Let $k_0 \in \mathbb{Z}$ satisfy $\delta^{k_0+1} \le r_0 < \delta^{k_0}$ and $C_5$ be a positive constant large enough, which is determined later. We formally write

$$\begin{aligned}
\Pi_2(a,g) &= \sum_{j\in\mathbb{Z}} \left\{ \sum_{\beta\in\mathscr{G}_j} \left(a, \psi_\beta^j\right) \psi_\beta^j \right\} \left\{ \sum_{\alpha\in\mathscr{A}_j} \left( [g - m_{B_0}(g)]\, \chi_{C_5 B_0}, \phi_\alpha^j \right) \phi_\alpha^j \right\} \\
&\quad + \sum_{j\in\mathbb{Z}} \left\{ \sum_{\beta\in\mathscr{G}_j} \left(a, \psi_\beta^j\right) \psi_\beta^j \right\} \left\{ \sum_{\alpha\in\mathscr{A}_j} \left\langle [g - m_{B_0}(g)]\, \chi_{\mathcal{X}\setminus(C_5 B_0)}, \phi_\alpha^j \right\rangle \phi_\alpha^j \right\} \\
&\quad + \sum_{j\in\mathbb{Z}} \left\{ \sum_{\beta\in\mathscr{G}_j} \left(a, \psi_\beta^j\right) \psi_\beta^j \right\} \left\{ \sum_{\alpha\in\mathscr{A}_j} \left\langle m_{B_0}(g), \phi_\alpha^j \right\rangle \phi_\alpha^j \right\} \\
&=: \Pi_2\left(a, [g - m_{B_0}(g)]\, \chi_{C_5 B_0}\right) + \Pi_2\left(a, [g - m_{B_0}(g)]\, \chi_{\mathcal{X}\setminus(C_5 B_0)}\right) \\
&\quad + \Pi_2\left(a, m_{B_0}(g)\right) \\
&=: \Pi_2^{(1)}(a,g) + \Pi_2^{(2)}(a,g) + \Pi_2^{(3)}(a,g),
\end{aligned}$$

where $m_{B_0}(g) := [\mu(B_0)]^{-1} \int_{B_0} g(x)\, d\mu(x)$.

By the proof of [27, Theorem 4.16], we know that $\Pi_2^{(1)}(a,g) \in H^1_{\mathrm{at}}(\mathcal{X}) \subset H^{\log}(\mathcal{X})$ and

$$\left\|\Pi_2^{(1)}(a,g)\right\|_{H^{\log}(\mathcal{X})} \lesssim \left\|\Pi_2^{(1)}(a,g)\right\|_{H^1_{\mathrm{at}}(\mathcal{X})} \lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

To estimate $\Pi_2^{(2)}(a,g)$, we first deal with $|([g - m_{B_0}(g)]\chi_{\mathcal{X}\setminus(C_5 B_0)}, \phi_\alpha^j)|$ for all $(j,\alpha) \in \mathscr{A}$ with $\mathscr{A}$ as in (4.9). Indeed, by (4.11), [2, Lemma 11.1] and (1.7), we conclude that

$$\begin{aligned}
(4.49)\quad & \left|\left([g - m_{B_0}(g)]\, \chi_{\mathcal{X}\setminus(C_5 B_0)}, \phi_\alpha^j\right)\right| \\
&\le \frac{1}{\sqrt{\mu_\alpha^j}} \int_{\mathcal{X}} |g(x) - m_{B_0}(g)|\, e^{-\nu\delta^{-j} d(x, x_\alpha^j)}\, d\mu(x) \\
&= \frac{1}{\sqrt{\mu_\alpha^j}} \int_{B(x_\alpha^j,\, \delta^j)} |g(x) - m_{B_0}(g)|\, d\mu(x) \\
&\quad + \frac{1}{\sqrt{\mu_\alpha^j}} \sum_{k=1}^\infty \int_{B(x_\alpha^j,\, 2^k\delta^j)\setminus B(x_\alpha^j,\, 2^{k-1}\delta^j)} |g(x) - m_{B_0}(g)|\, e^{-\nu\delta^{-j} d(x, x_\alpha^j)}\, d\mu(x) \\
&\lesssim \sqrt{\mu_\alpha^j} \|g\|_{\mathrm{BMO}\,(\mathcal{X})} + \frac{1}{\sqrt{\mu_\alpha^j}} \sum_{k=1}^\infty \int_{B(x_\alpha^j,\, 2^k\delta^j)} |g(x) - m_{B_0}(g)|\, e^{-\nu 2^{k-1}}\, d\mu(x) \\
&\lesssim \sqrt{\mu_\alpha^j} \|g\|_{\mathrm{BMO}\,(\mathcal{X})} + \frac{1}{\sqrt{\mu_\alpha^j}} \sum_{k=1}^\infty \int_{B(x_\alpha^j,\, 2^k\delta^j)} \left[ \left| g(x) - m_{B(x_\alpha^j,\, 2^k\delta^j)}(g) \right| \right. \\
&\quad \left. + \sqrt{\mu_\alpha^j} \left| m_{B(x_\alpha^j,\, 2^k\delta^j)}(g) - m_{B_0}(g) \right| \right] e^{-\nu 2^{k-1}}\, d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sqrt{\mu_\alpha^j}\|g\|_{\mathrm{BMO}\,(\mathcal{X})}\\
&\quad + \sqrt{\mu_\alpha^j}\sum_{k=1}^\infty e^{-\nu 2^{k-1}}2^{kn}\|g\|_{\mathrm{BMO}\,(\mathcal{X})}\left[1+\log\frac{\delta^j+r_0+d(x_\alpha^j,x_0)}{\min\{\delta^j,r_0\}}\right]\\
&\lesssim \sqrt{\mu_\alpha^j}\|g\|_{\mathrm{BMO}\,(\mathcal{X})}\left[1+\log\frac{\delta^j+r_0+d(x_\alpha^j,x_0)}{\min\{\delta^j,r_0\}}\right].
\end{aligned}
$$

By (4.45), we know that, for any $j\in\mathbb{Z}$, $\alpha\in\mathscr{A}_j$ with $\mathscr{A}_j$ as in (4.6), and $\beta\in\mathscr{G}_j$ with $\mathscr{G}_j$ as in (4.13), $a^j_{\alpha,\beta}$ is a $(1,2)$-atom, multiplied by a positive harmless constant, supported in $B(x_\alpha^j,10\delta^j)$, and hence

$$
\begin{aligned}
\mathrm{I} :=&\sum_{j\in\mathbb{Z}}\sum_{\alpha\in\mathscr{A}_j}\sum_{\beta\in\mathscr{G}_j}\left|\left([g-m_{B_0}(g)]\,\chi_{\mathcal{X}\setminus(C_5B_0)},\phi_\alpha^j\right)\right|\left|\left(a,\psi_\beta^j\right)\right|\left\|\phi_\alpha^j\psi_\beta^j\right\|_{H^1_{\mathrm{at}}(\mathcal{X})}\\
\lesssim&\sum_{j\in\mathbb{Z}}\sum_{\alpha\in\mathscr{A}_j}\sum_{\beta\in\mathscr{G}_j}\left|\left([g-m_{B_0}(g)]\,\chi_{\mathcal{X}\setminus(C_5B_0)},\phi_\alpha^j\right)\right|\left|\left(a,\psi_\beta^j\right)\right|e^{-\frac{\nu}{2}\delta^{-j}d(x_\alpha^j,y_\beta^j)}\\
\sim&\sum_{j=k_0+1}^{\infty}\sum_{\alpha\in\mathscr{A}_j}\sum_{\beta\in\mathscr{G}_j}\left|\left([g-m_{B_0}(g)]\,\chi_{\mathcal{X}\setminus(C_5B_0)},\phi_\alpha^j\right)\right|\left|\left(a,\psi_\beta^j\right)\right|e^{-\frac{\nu}{2}\delta^{-j}d(x_\alpha^j,y_\beta^j)}\\
&+\sum_{j=-\infty}^{k_0}\cdots .
\end{aligned}
$$

From (4.49) and the proof of [27, Theorem 4.16], via choosing $C_5$ to be largely enough, we deduce that

$$\mathrm{I}\lesssim\|g\|_{\mathrm{BMO}\,(\mathcal{X})},$$

$\Pi_2^{(2)}(a,g)\in H^1_{\mathrm{at}}(\mathcal{X})\subset H^{\log}(\mathcal{X})$ and

$$\left\|\Pi_2^{(2)}(a,g)\right\|_{H^{\log}(\mathcal{X})}\lesssim\left\|\Pi_2^{(2)}(a,g)\right\|_{H^1_{\mathrm{at}}(\mathcal{X})}\lesssim\mathrm{I}\lesssim\|g\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

Finally, we estimate $\Pi_2^{(3)}(a,g)$. From [2, Lemma 10.1], it follows that, for all $j\in\mathbb{Z}$,

$$P_j1=1. \tag{4.50}$$

From (4.50), $a\in L^2(\mathcal{X})$ and Theorem 4.33, it follows that

$$\Pi_2^{(3)}(a,g)=m_{B_0}(g)\Pi_2(a,1)=m_{B_0}(g)a.$$

By this, Remark 4.6, [49, Proposition 3.2(ii)] and [49, Lemma 3.2], we have

$$
\begin{aligned}
\left\|\Pi_2^{(3)}(a,g)\right\|_{H^{\log}(\mathcal{X})}&\lesssim\||m_{B_0}(g)-g|\,\mathcal{M}(a)\|_{L^{\log}(\mathcal{X})}+\||g|\mathcal{M}(a)\|_{L^{\log}(\mathcal{X})}\\
&\lesssim\||m_{B_0}(g)-g|\,\mathcal{M}(a)\|_{L^1(\mathcal{X})}+\|a\|_{L^1(\mathcal{X})}\||g|\|_{\mathrm{BMO}^+(\mathcal{X})}\\
&\lesssim\|g\|_{\mathrm{BMO}\,(\mathcal{X})}+\|g\|_{\mathrm{BMO}^+(\mathcal{X})}\lesssim\|g\|_{\mathrm{BMO}^+(\mathcal{X})},
\end{aligned}
$$

which, together with the estimates of $\Pi_2^{(1)}(a,g)$ and $\Pi_2^{(2)}(a,g)$, implies that $\Pi_2(a,g)$ belongs to $H^{\log}(\mathcal{X})$ and (4.48) holds true.

By the above proof of (4.48), we conclude that there exists

$$h := \widetilde{\Pi}_2(a,g) = \Pi_2^{(1)}(a,g) + \Pi_2^{(2)}(a,g) \in H^1_{\rm at}(\mathcal{X})$$

satisfying that $\|h\|_{H^1_{\rm at}(\mathcal{X})} \lesssim \|g\|_{\mathrm{BMO}\,(\mathcal{X})}$ and

$$\Pi_2(a,g) = h + am_{B_0}(g),$$

which, combined with some arguments used in the proof [27, Theorem 4.16], further implies that $\Pi_2$ can be extended to a bounded bilinear operator from $H^1_{\rm at}(\mathcal{X}) \times \mathrm{BMO}\,(\mathcal{X})$ into $H^{\log}(\mathcal{X})$. This finishes the proof of Lemma 4.44. □

**Remark 4.45.** Using $\Pi_2(f,g) = \Pi_1(g,f)$ for all $f$, $g \in L^2(\mathcal{X})$ and Lemma 4.43, we conclude that $\Pi_2$ as in (4.24) can also be extended to a bounded bilinear operator from $L^2(\mathcal{X}) \times L^2(\mathcal{X})$ into $H^1_{\rm at}(\mathcal{X})$.

Before proving Theorem 4.9, we recall the result from [27, Theorem 4.9] on the boundedness of $\Pi_3$ as in (4.24). We first formally write

$$\Pi_3(a,g) := \sum_{j\in\mathbb{Z}} \left[\sum_{\beta\in\mathscr{G}_j} \left(a, \psi^j_\beta\right) \psi^j_\beta\right] \left[\sum_{\gamma\in\mathscr{G}_j} \langle g, \psi^j_\gamma\rangle \psi^j_\gamma\right] \tag{4.51}$$

for any $(1,2)$-atom $a$ and $g \in \mathrm{BMO}\,(\mathcal{X})$, where $\mathscr{G}_j$ for any $j \in \mathbb{Z}$ is as in (4.13). We point out that, if $a$, $g \in L^2(\mathcal{X})$, then $\Pi_3(a,g)$ in (4.51) coincides with $\Pi_3(a,g)$ in (4.24) with $f$ replaced by $a$ and, in this case, it is known that $\Pi_3(a,g) \in L^1(\mathcal{X})$ (see [27, Lemma 3.3]).

**Lemma 4.46** ([27])**.** *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. Then, for any $(1,2)$-atom $a$ and $g \in \mathrm{BMO}\,(\mathcal{X})$, $\Pi_3(a,g)$ in (4.51) belongs to $L^1(\mathcal{X})$ and $\Pi_3$ can be extended to a bounded bilinear operator from $H^1_{\rm at}(\mathcal{X}) \times \mathrm{BMO}\,(\mathcal{X})$ into $L^1(\mathcal{X})$.*

Now we are ready to present the proof of Theorem 4.9.

*Proof of Theorem 4.9.* We first claim that, to prove Theorem 4.9, it suffices to show that, for any $(1,2)$-atom $a$, supported in a ball $B_0 := B(x_0, r_0)$ with some $x_0 \in \mathcal{X}$ and $r_0 \in (0,\infty)$, and $g \in \mathrm{BMO}\,(\mathcal{X})$,

$$a \times g = \sum_{i=1}^3 \Pi_i(a,g) \quad \text{in} \quad (\mathcal{G}^\epsilon_0(\varrho,\vartheta))', \tag{4.52}$$

where $\epsilon$, $\varrho$ and $\vartheta$ are as in Theorem 4.9.

Assuming that (4.52) holds true, we now show Theorem 4.9. Indeed, for any $f \in H^1_{\rm at}(\mathcal{X})$, from Definition 4.4, it follows that there exist a sequence $\{a_j\}_{j\in\mathbb{N}}$ of $(1,2)$-atoms and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $H^1_{\rm at}(\mathcal{X})$. For any $N \in \mathbb{N}$, let $f_N := \sum_{j=1}^N \lambda_j a_j$. We then obtain

$$\lim_{N\to\infty} f_N = f \quad \text{in} \quad H^1_{\rm at}(\mathcal{X}). \tag{4.53}$$

By (4.52), we know that

$$f_N \times g = \Pi_1(f_N, g) + \Pi_2(f_N, g) + \Pi_3(f_N, g) \quad \text{in} \quad (\mathcal{G}^\epsilon_0(\varrho,\vartheta))'. \tag{4.54}$$

We now prove that

$$\lim_{N\to\infty} f_N \times g = f \times g \quad \text{in} \quad (\mathcal{G}_0^\epsilon(\varrho,\vartheta))'. \tag{4.55}$$

To this end, for any $h \in \mathcal{G}_0^\epsilon(\varrho,\vartheta)$, from Lemma 4.8 and (4.53), we deduce that

$$\begin{aligned}|\langle f_N \times g, h\rangle - \langle f \times g, h\rangle| &= |\langle gh, f_N - f\rangle| \le \|gh\|_{\mathrm{BMO}\,(\mathcal{X})} \|f_N - f\|_{H^1_{\mathrm{at}}(\mathcal{X})} \\ &\lesssim \frac{1}{V_1(x_1)} \|h\|_{\mathcal{G}(\varrho,\vartheta)} \|g\|_{\mathrm{BMO}^+(\mathcal{X})} \|f_N - f\|_{H^1_{\mathrm{at}}(\mathcal{X})} \to 0\end{aligned}$$

as $N \to \infty$, which proves (4.55).

Moreover, from (4.53), Lemmas 4.37, 4.44 and 4.46, we deduce that

$$\lim_{N\to\infty} \Pi_3(f_N, g) = \Pi_3(f,g) \quad \text{in} \quad L^1(\mathcal{X}),$$

$\lim_{N\to\infty} \Pi_1(f_N, g) = \Pi_1(f,g)$ in $H^1_{\mathrm{at}}(\mathcal{X})$, and $\lim_{N\to\infty} \Pi_2(f_N,g) = \Pi_2(f,g)$ in $H^{\log}(\mathcal{X})$, which immediately imply that all of them hold true in $(\mathcal{G}_0^\epsilon(\varrho,\vartheta))'$. By these facts, (4.53), (4.54) and (4.55), we obtain

$$\begin{aligned}f \times g = \lim_{N\to\infty} f_N \times g &= \lim_{N\to\infty} \sum_{i=1}^3 \Pi_i\,(f_N, g) \\ &= \sum_{i=1}^3 \Pi_i(f,g) \quad \text{in} \quad (\mathcal{G}_0^\epsilon(\varrho,\vartheta))',\end{aligned}$$

which, combined with Lemmas 4.37, 4.44 and 4.46, then completes the proof of Theorem 4.9 with $\mathscr{L} := \Pi_3$ and $\mathscr{H} := \Pi_1 + \Pi_2$.

Now we show (4.52). From Theorem 4.34 and Remark 4.35, we deduce that

$$\widetilde{g} := \sum_{j\in\mathbb{Z}} \sum_{\beta\in\mathscr{G}_j} \left\langle g, \psi_\beta^j \right\rangle \left[\psi_\beta^j - \chi_{\{k\in\mathbb{Z}:\ \delta^k > r_1\}}(j)\psi_\beta^j(x_1)\right] \tag{4.56}$$

converges in both $L^2_{\mathrm{loc}}(\mathcal{X})$ and $\mathrm{BMO}\,(\mathcal{X})$.

Now we choose a fixed collection $\{\mathscr{C}_N :\ N \in \mathbb{N},\ \mathscr{C}_N \subset \mathscr{C}$ and $\mathscr{C}_\mathrm{N}$ is finite$\}$ as in (4.22) and let

$$\widetilde{g}_N := \sum_{(j,\,\beta)\in\mathscr{C}_N} \left\langle g, \psi_\beta^j \right\rangle \left[\psi_\beta^j - \chi_{\{k\in\mathbb{Z}:\ \delta^k > r_1\}}(j)\psi_\beta^j(x_1)\right] = \sum_{(j,\,\beta)\in\mathscr{C}_N} \left\langle g, \psi_\beta^j \right\rangle \psi_\beta^j =: g_N$$

in $\mathrm{BMO}\,(\mathcal{X})$.

From the finiteness of $\mathscr{C}_N$, it follows that $g_N \in L^2(\mathcal{X})$, which, together with Lemmas 4.36 and 4.43, [27, lemma 3.3] and Remark 4.45, further implies that, for any $N \in \mathbb{N}$,

$$a g_N = \sum_{i=1}^3 \Pi_1\,(a, g_N) \quad \text{in} \quad L^1(\mathcal{X}). \tag{4.57}$$

Then we claim that, for any $h \in \mathcal{G}_0^\epsilon(\varrho,\vartheta)$, $\lim_{N\to\infty}\langle a \times \widetilde{g}_N, h\rangle = \langle a \times \widetilde{g}, h\rangle$. Indeed, by the definition of the distribution, the duality between $H^1_{\mathrm{at}}(\mathcal{X})$ and $\mathrm{BMO}\,(\mathcal{X})$, Lemma 4.8 and (4.56), we conclude that

$$|\langle a \times \widetilde{g}_N, h\rangle - \langle a \times \widetilde{g}, h\rangle| \tag{4.58}$$

$$
\begin{aligned}
&= |\langle (\widetilde{g}_N - \widetilde{g})\, h, a\rangle| \le \|(\widetilde{g}_N - \widetilde{g})\, h\|_{\mathrm{BMO}\,(\mathcal{X})} \|a\|_{H^1_{\mathrm{at}}(\mathcal{X})} \\
&\lesssim \frac{1}{V_1(x_1)} \|h\|_{\mathcal{G}(\varrho,\vartheta)} \|\widetilde{g}_N - \widetilde{g}\|_{\mathrm{BMO}^+(\mathcal{X})} \\
&\lesssim \frac{1}{V_1(x_1)} \|h\|_{\mathcal{G}(\varrho,\vartheta)} \\
&\quad \times \left[ \|\widetilde{g}_N - \widetilde{g}\|_{\mathrm{BMO}\,(\mathcal{X})} + \frac{1}{\sqrt{V_1(x_1)}} \left\| (\widetilde{g}_N - \widetilde{g})\, \chi_{B(x_1,1)} \right\|_{L^2(\mathcal{X})} \right] \\
&\to 0 \quad \text{as} \quad N \to \infty.
\end{aligned}
$$

This shows the above claim.

It is shown in Remark 4.35 that $g - \widetilde{g} =: c_4$ is a constant. Let $c_{(N)} := \widetilde{g}_N - g_N$ for any $N \in \mathbb{N}$. It is easy to see that $c_{(N)}$ is a constant, depending on $N$, for each $N \in \mathbb{N}$. From this, (4.58), (4.56), (4.57), Lemmas 4.37, 4.44 and 4.46, $\Pi_2(a,1) = a$ and (4.19), we deduce that

$$
\begin{aligned}
a \times g &= a \times \widetilde{g} + c_4 a = \lim_{N\to\infty} a \times \widetilde{g}_N + c_4 a = \lim_{N\to\infty} \left[ a g_N + c_{(N)} a \right] + c_4 a \\
&= \lim_{N\to\infty} \left[ \sum_{i=1}^{3} \Pi_i\,(a, g_N) + c_{(N)} \Pi_2(a,1) \right] + c_4 a = \lim_{N\to\infty} \sum_{i=1}^{3} \Pi_i\,(a, \widetilde{g}_N) + c_4 a \\
&= \Pi_1\,(a, \widetilde{g}) + [\Pi_2\,(a, \widetilde{g}) + c_4 \Pi_2(a,1)] + \Pi_3\,(a, \widetilde{g}) \\
&= \sum_{i=1}^{3} \Pi_i\,(a, g) \quad \text{in} \quad (\mathcal{G}_0^\epsilon(\varrho,\vartheta))',
\end{aligned}
$$

which completes the proof of (4.52) and hence Theorem 4.9. □

## 5. Further remarks

In this section, we list some unsolved problems on bilinear decompositions for products of functions in Hardy spaces and Lipschitz spaces and their applications on spaces of homogeneous type.

For any $p \in (0,1)$, $x \in \mathcal{X}$ and $t \in [0,\infty)$, let $H^p_{\mathrm{at}}(\mathcal{X})$ and $\mathrm{Lip}_{1/p-1}(\mathcal{X})$ be the Hardy space and the Lipschitz space introduced in [14], respectively, $\varphi_p$ some Musielak-Orlicz functions, corresponding to the Musielak-Orlicz-type space $L^{\varphi_p}(\mathcal{X})$ and Musielak-Orlicz-type Hardy space $H^{\varphi_p}(\mathcal{X})$.

The following problem is natural extensions of Theorem 4.9.

**Problem 5.1.** Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and $p \in (\frac{n}{n+1}, 1)$. Prove that there exist two bounded bilinear operators $\mathscr{L}: H^p_{\mathrm{at}}(\mathcal{X}) \times \mathrm{Lip}_{1/p-1}(\mathcal{X}) \to L^1(\mathcal{X})$ and $\mathscr{H}: H^p_{\mathrm{at}}(\mathcal{X}) \times \mathrm{Lip}_{1/p-1}(\mathcal{X}) \to H^{\varphi_p}(\mathcal{X})$ such that, for all $f \in H^p_{\mathrm{at}}(\mathcal{X})$ and $g \in \mathrm{Lip}_{1/p-1}(\mathcal{X})$,

$$
f \times g = \mathscr{L}(f,g) + \mathscr{H}(f,g) \quad \text{in} \quad (\mathcal{G}_0^\epsilon(\varrho,\vartheta))',
$$

where $\epsilon \in (0,1]$ and $\varrho$, $\vartheta \in (0,\epsilon]$.

Problems (5.1) is also not clear even on Euclidean spaces.

Now we introduce the space $H^p_g(\mathcal{X})$, which is a variant of [46, Definition 2.2].

**Definition 5.2.** Let $p \in (0,1]$, $g$ be a non-constant $\mathrm{Lip}_{1/p-1}(\mathcal{X})$-function. A function $f$ in $H^p_{\rm at}(\mathcal{X})$ is said to belong to the space $H^p_g(\mathcal{X})$ if $[g,\mathcal{M}](f)$, defined by setting

$$[g,\mathcal{M}](f)(x) := \mathcal{M}\left(g(x)f(\cdot) - g(\cdot)f(\cdot)\right)(x), \quad \forall\, x \in \mathcal{X},$$

belongs to $L^1(\mathcal{X})$, where $\mathcal{M}$ is as in (4.2). Moreover, the $H^p_g(\mathcal{X})$-norm of $f$ is denoted by

$$\|f\|_{H^p_g(\mathcal{X})} := \|f\|_{H^p_{\rm at}(\mathcal{X})}\|g\|_{\mathrm{Lip}_{1/p-1}(\mathcal{X})} + \|[g,\mathcal{M}](f)\|_{L^1(\mathcal{X})}.$$

Then we give the second open question of this article, which is an extension of [46, Theorem 1.3].

**Problem 5.3.** Let $p \in (\frac{n}{n+1}, 1]$, $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, $g$ a non-constant $\mathrm{Lip}_{1/p-1}(\mathcal{X})$-function when $p \in (\frac{n}{n+1}, 1)$ or $\mathrm{BMO}\,(\mathcal{X})$-function when $p = 1$, and $T$ a Calderón-Zygmund operator, which is bounded on $L^2(\mathcal{X})$, satisfying $T^*1 = T^*g = 0$. Prove that the commutator $[b, T]$ maps continuously from $H^p_g(\mathcal{X})$ into $H^p_{\rm at}(\mathcal{X})$.

Problem 5.3 is unknown even on Euclidean spaces.

In order to introduce the third open problem, let $\alpha := n(1/p-1)$, $H^p_\rho(\mathcal{X})$ and $\mathrm{Lip}_{\alpha,\rho}(\mathcal{X})$ be, respectively, the local Hardy space and local Lipschitz space from [68] with

$$\mathcal{D} := \{B(x,r) \subset \mathcal{X} :\ x \in \mathcal{X},\ r \geq \rho(x)\}.$$

Let $\varphi_{p,\rho}$ be some Musielak-Orlicz type function, which corresponds to the Musielak-Orlicz-type Hardy space $H^{\varphi_{p,\rho}}(\mathcal{X})$.

Now we are ready to state the third open problems of this section, which generalizes Theorem 4.21 and Problem 5.1.

**Problem 5.4.** Let $p \in (\frac{n}{n+1}, 1)$, $\alpha := 1/p - 1$, $(\mathcal{X}, d, \mu)$ be a space of homogeneous type satisfying the additional Assumption 4.14 and $\rho$ an admissible function as in Definition 4.12. Prove that there exist two bounded bilinear operators $\mathscr{L}_\rho$ : $H^p_\rho(\mathcal{X}) \times \mathrm{Lip}_{\alpha,\rho}(\mathcal{X}) \to L^1(\mathcal{X})$ and $\mathscr{H}_\rho :\ H^p_\rho(\mathcal{X}) \times \mathrm{Lip}_{\alpha,\rho}(\mathcal{X}) \to H^{\varphi_{p,\rho}}(\mathcal{X})$ such that, for all $f \in H^p_\rho(\mathcal{X})$ and $g \in \mathrm{Lip}_{\alpha,\rho}(\mathcal{X})$,

$$f \times g = \mathscr{L}_\rho(f,g) + \mathscr{H}_\rho(f,g) \quad \text{in} \quad (\mathcal{G}^\epsilon_0(\varrho,\vartheta))',$$

where $\epsilon \in (0,1)$ and $\varrho$, $\vartheta \in (0,\epsilon]$.

**Remark 5.5.**
- (i) Let $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)$ be the Euclidean space equipped with the $D$-dimensional Lebesgue measure $dx$ and $\rho \equiv 1$. Then $H^p_1(\mathbb{R}^D)$ coincides with $h^p(\mathbb{R}^D)$ and Problem 5.4 generalizes Theorem 3.1.
- (ii) Problem 5.4 is still unclear even on Euclidean spaces.
- (iii) The applications of bilinear decompositions in Theorem 4.21 and Problem 5.4 to the endpoint boundedness of commutators associated to the admissible function $\rho$ are also possible, whose explicit forms are more difficult to predict, the details being omitted.

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