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STRUCTURE OF APPROXIMATE SOLUTIONS OF AUTONOMOUS VARIATIONAL PROBLEMS

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Abstract. In this paper we study the structure of approximate solutions of autonomous variational problems on large finite intervals. Our goal is to show that approximate solutions are determined mainly by the integrand, and are essentially independent of the choice of time interval and data. In the first part of the paper we discuss our recent results on Lagrange problems. The second part of the paper contains new results on the structure of approximate solutions of Bolza problems.

1. Introduction

The study of variational and optimal control problems defined on infinite (large) intervals has recently been a rapidly growing area of research [2, 5–16, 18, 19, 22, 23, 25, 27–38]. These problems arise in engineering [1, 10, 30, 39], in models of economic growth [10, 21, 24, 30], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 26] and in the theory of thermodynamical equilibrium for materials [17, 20].

In this paper we analyze the structure of approximate solutions of the Lagrange variational problems

$$(P_1) \qquad \int_0^T f(z(t), z'(t))dt \to \min,\ z(0) = x,\ z(T) = y,$$

$$z:\ [0, T] \to R^n \text{ is an absolutely continuous (a. c.) function,}$$

$$(P_2) \qquad \int_0^T f(z(t), z'(t))dt \to \min,\ z(0) = x,$$

$$z:\ [0, T] \to R^n \text{ is an a. c. function,}$$

$$(P_3) \qquad \int_0^T f(z(t), z'(t))dt \to \min,\ z:\ [0, T] \to R^n \text{ is an a. c. function}$$

and Bolza variational problems

$$(P_4) \qquad \int_0^T f(z(t), z'(t))dt + h(z(T)) \to \min,\ z(0) = x,$$

$$z:\ [0, T] \to R^n \text{ is an a. c. function,}$$

$$(P_5) \qquad \int_0^T f(z(t), z'(t))dt + h(z(T)) + \xi(z(0)) \to \min,$$

$$z:\ [0, T] \to R^n \text{ is an a. c. function,}$$

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where $T > 0$ is sufficiently large, $x, y \in R^n$, $f : R^n \times R^n \to R^1$ is an integrand and $h, \xi : R^n \times R^n \to R^1$ belong to a space of functions described below.

In our research which was summarized in [30] we were interested in turnpike properties of the approximate solutions of problem (P_1) which are independent of the length of the interval, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand, and are essentially independent of the choice of an interval and endpoint conditions, except in regions close to the endpoints of the time interval.

It is clear that an optimal solution $v : [0, T] \to R^n$ of the variational problem (P_1) always depends on the integrand f and on x, y, T.

We say that the integrand f has the turnpike property if for any $\epsilon > 0$ there exist constants $L_1 > L_2 > 0$ which depend only on $|x|$, $|y|$, ϵ such that for each $\tau \in [L_1, T - L_1]$ the set $\{v(t) : t \in [\tau, \tau + L_2]\}$ is equal to a set $H(f)$ up to ϵ in the Hausdorff metric where $H(f) \subset R^n$ is a compact set depending only on the integrand f.

Thus if the integrand f has the turnpike property, then for large enough T the dependence on x, y, T is not essential. In [29] this turnpike property was established for a certain large class of integrands.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [24]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics [21,30]. Many turnpike results can be found in [30].

In [36] we studied the structure of approximate solutions of Lagrange problems (P_2) and (P_3) in regions close to the endpoints of the time intervals. We showed that in regions close to the right endpoint T of the time interval these approximate solutions are determined only by the integrand, and are essentially independent of the choice of the interval and the endpoint value x. For problems (P_3), approximate solutions are determined only by the integrand also in regions close to the left endpoint 0 of the time interval.

More precisely, in [36] we define $\bar{f}(x, y) = f(x, -y)$ for all $x, y \in R^n$ and consider the set $\mathcal{P}(\bar{f})$ of all solutions of a corresponding infinite horizon variational problem associated with the integrand $\bar{f}$. For given positive constants ϵ, τ, we show that if T is large enough and $v : [0, T] \to R^n$ is an approximate of the variational problem (P_2), then $|v(T - t) - w(t)| \leq \epsilon$ for all $t \in [0, \tau]$, where $w \in \mathcal{P}(\bar{f})$. Moreover, using the Baire category approach, we showed that for most integrands f the set $\mathcal{P}(\bar{f})$ is a singleton.

In the first part of the paper we discuss our results on the structure of approximate solutions of Lagrange problems $(P_1) - (P_3)$ obtained in [27,29,36]. The second part of the paper contains new results on the structure of approximate solutions of Bolza problems (P_4) and (P_5).

Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to \infty$ as $t \to \infty$.

Denote by $\mathcal{A}$ the set of all continuous functions $f: R^n \times R^n \to R^1$ which satisfy the following assumptions:

A(i) for each $x \in R^n$ the function $f(x, \cdot): R^n \to R^1$ is convex;

A(ii) $f(x,u) \ge \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x,u) \in R^n \times R^n$;

A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(x_1,u_1) - f(x_2,u_2)| \le \epsilon \max\{f(x_1,u_1), f(x_2,u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M,\ i = 1,2,\ |u_i| \ge \Gamma,\ i = 1,2, \quad |x_1 - x_2|,\ |u_1 - u_2| \le \delta.$$

It is easy to show that an integrand $f = f(x,u) \in C^1(R^{2n})$ belongs to $\mathcal{A}$ if f satisfies assumptions A(i), A(ii) and if there exists an increasing function $\psi_0 : [0,\infty) \to [0,\infty)$ such that

$$\max\{|\partial f/\partial x(x,u)|,\ |\partial f/\partial u(x,u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $x, u \in R^n$. Here $\partial f/\partial x = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ and $\partial f/\partial u = (\partial f/\partial u_1, \ldots, \partial f/\partial u_n)$.

For the set $\mathcal{A}$ we consider the uniformity which is determined by the following base:

$$\begin{aligned} E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{A}\times\mathcal{A} :\ &|f(x,u) - g(x,u)| \le \epsilon \\ &\text{for all } u, x \in R^n \text{ satisfying } |x|, |u| \le N\} \\ \cap\{(f,g) \in \mathcal{A}\times\mathcal{A} :\ &(|f(x,u)|+1)(|g(x,u)|+1)^{-1} \in [\lambda^{-1}, \lambda] \\ &\text{for all } x, u \in R^n \text{ satisfying } |x| \le N\}, \end{aligned}$$

where $N, \epsilon > 0$ and $\lambda > 1$. It is known [30] that the uniform space $\mathcal{A}$ is metrizable and complete.

We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt \tag{1.1}$$

where $f \in \mathcal{A}$, $-\infty < T_1 < T_2 < \infty$ and $x: [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathcal{A}$, $y, z \in R^n$ and real numbers T_1, T_2 satisfying $T_1 < T_2$ we set

$$\begin{aligned} U^f(T_1,T_2,y,z) = \inf\{I^f(T_1,T_2,x) :\ x :\ [T_1,T_2] \to R^n \\ \text{is an a.c. function satisfying } x(T_1) = y,\ x(T_2) = z\}. \end{aligned} \tag{1.2}$$

It is easy to see that $-\infty < U^f(T_1,T_2,y,z) < \infty$ for each $f \in \mathcal{A}$, each $y, z \in R^n$ and all numbers T_1, T_2 satisfying $T_1 < T_2$.

A function $x(\cdot)$ defined on unbounded interval with the values in a finite–dimensional Euclidean space is called absolutely continuous (a. c.) if it is absolutely continuous on any bounded subinterval of its domain.

Let $f \in \mathcal{A}$. For any a.c. function $v : [0,\infty) \to R^n$ we set

$$J(v) = \liminf_{T\to\infty} T^{-1} I^f(0,T,v). \tag{1.3}$$

The real number

$$\mu(f) = \inf\{J(v) :\ v : [0,\infty) \to R^n \text{ is an a.c. function}\} \tag{1.4}$$

is called the minimal long-run average cost growth rate of f.

Clearly, $-\infty < \mu(f) < \infty$. By Theorems 3.6.1 and 3.6.2 of [30],

$$U^f(0,T,x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x,y) \tag{1.5}$$

for all $x, y \in R^n$ and all $T \in (0,\infty)$, where $\pi^f : R^n \to R^1$ is a continuous function and

$$(T,x,y) \to \theta^f_T(x,y) \in R^1 \text{ is a continuous nonnegative function} \tag{1.6}$$

defined for all $T > 0$ and all $x, y \in R^n$,

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,v) - \mu(f)T] : \ v : [0,\infty) \to R^n$$

$$\text{is an a.c. function satisfying } v(0) = x\}, \ x \in R^n \tag{1.7}$$

and

$$\text{for every } T > 0, \text{ every } x \in R^n \text{ there is } y \in R^n \text{ satisfying } \theta^f_T(x,y) = 0. \tag{1.8}$$

An a.c. function $x : [0,\infty) \to R^n$ is called (f)-good if the function

$$T \to I^f(0,T,x) - \mu(f)T, \ T \in (0,\infty)$$

is bounded.

By Theorem 3.6.3 of [30], for each $f \in \mathcal{A}$ and each $z \in R^n$ there exists an (f)-good function $v : [0,\infty) \to R^n$ satisfying $v(0) = z$.

In the sequel we use the following result (Proposition 4.1.1 of [30]).

Proposition 1.1. *For any a.c. function $x : [0,\infty) \to R^n$ either $I^f(0,T,x) - T\mu(f) \to \infty$ as $T \to \infty$ or*

$$\sup\{|I^f(0,T,x) - T\mu(f)| : \ T \in (0,\infty)\} < \infty.$$

Moreover any (f)-good function $x : [0,\infty) \to R^n$ is bounded.

We denote $d(x,B) = \inf\{|x - y| : \ y \in B\}$ for $x \in R^n$ and $B \subset R^n$ and denote by $\mathrm{dist}(A,B)$ the distance in the Hausdorff metric for two sets $A \subset R^n$ and $B \subset R^n$. For every bounded a. c. function $x : [0,\infty) \to R^n$ define

$$\Omega(x) = \{y \in R^n : \ \text{ there exists a sequence } \{t_i\}_{i=1}^\infty \subset (0,\infty)$$

$$\text{for which } t_i \to \infty, \ x(t_i) \to y \text{ as } i \to \infty\}. \tag{1.9}$$

We say that an integrand $f \in \mathcal{A}$ has an asymptotic turnpike property, or briefly (ATP), if $\Omega(v_2) = \Omega(v_1)$ for all (f)-good functions $v_i : [0,\infty) \to R^n$, $i = 1,2$.

By Theorem 3.1.1 of [30], there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ posseses (ATP). In other words, (ATP) holds for a typical (generic) integrand $f \in \mathcal{A}$.

By Proposition 1.1 for each integrand $f \in \mathcal{A}$ which posseses (ATP) there exists a compact set $H(f) \subset R^n$ such that $\Omega(v) = H(f)$ for each (f)-good function $v : [0,\infty) \to R^n$. In this case we say that the set $H(f)$ is the turnpike of f.

The following turnpike result was obtained in [27]. For its proof see also Theorem 3.1.4 of [30].

Theorem 1.2. *Assume that an integrand $f \in \mathcal{A}$ has the asymptotic turnpike property and that $M_0, M_1, \epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $l, S > 0$ and integers $L, Q_* \geq 1$ such that for each $g \in \mathcal{U}$, each pair of numbers $T_1 \geq 0$, $T_2 \geq T_1 + L + lQ_*$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies*

$$|v(T_i)| \leq M_1, \ i = 1, 2,$$

$$I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + M_0$$

the inequality $|v(t)| \leq S$ holds for all $t \in [T_1, T_2]$ and there exist sequences of numbers $\{b_i\}_{i=1}^Q$, $\{c_i\}_{i=1}^Q \subset [T_1, T_2]$ such that

$$Q \leq Q_*, \ 0 \leq c_i - b_i \leq l, \ i = 1, \dots, Q$$

and that

$$dist(H(f), \{v(t) : \ t \in [T, T + L]\}) \leq \epsilon$$

for each $T \in [T_1, T_2 - L] \setminus \cup_{i=1}^Q [b_i, c_i]$.

Denote by $\mathcal{M}$ the set of all functions $f \in C^1(R^{2n})$ which satisfy the following assumptions:

$$\partial f / \partial u_i \in C^1(R^{2n}) \text{ for } i = 1, \dots, n;$$

the matrix $(\partial^2 f / \partial u_i \partial u_j)(x, u)$, $i, j = 1, \dots, n$ is positive definite for all $(x, u) \in R^{2n}$;

$$f(x, u) \geq \max\{\psi(|x|), \ \psi(|u|)|u|\} - a \text{ for all } (x, u) \in R^n \times R^n;$$

there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i : [0, \infty) \to [0, \infty)$, $i = 0, 1, 2$ such that

$$\phi_0(t)/t \to \infty \text{ as } t \to \infty,$$

$$f(x, u) \geq \phi_0(c_0|u|) - \phi_1(|x|), \ x, u \in R^n,$$

$$\max\{|\partial f / \partial x_i(x, u)|, \ |\partial f / \partial u_i(x, u)|\} \leq \phi_2(|x|)(1 + \phi_0(|u|)),$$

$$x, u \in R^n, \ i = 1, \dots, n.$$

It is easy to see that $\mathcal{M} \subset \mathcal{A}$.

In [29] we established the following result which shows that for an integrand $f \in \mathcal{M}$, (ATP) implies the turnpike property described above with the turnpike $H(f)$ (for its proof see also Theorem 5.1.1 of [30]).

Theorem 1.3. *Assume that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property and that $\epsilon, K > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq K, \ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta$$

the inequality $|v(t)| \leq M$ holds for all $t \in [0, T]$ and

$$dist(H(f), \{v(t) : \ t \in [\tau, \tau + l]\}) \leq \epsilon \quad (1.10)$$

for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (1.10) holds for each $\tau \in [0, T - l_0]$ and if $d(v(T), H(f)) \leq \delta$, then (1.10) holds for each $\tau \in [l_0, T - l]$.

Let $k \geq 1$ be an integer. Denote by $\mathcal{A}_k$ the set of all integrands $f \in \mathcal{A} \cap C^k(R^{2n})$. For any $p = (p_1, \dots, p_{2n}) \in \{0, \dots, k\}^{2n}$ set $|p| = \sum_{i=1}^{2n} p_i$. For each $f \in C^k(R^{2n})$ and each $p = (p_1, \dots, p_{2n}) \in \{0, \dots, k\}^{2n}$ satisfying $|p| \leq k$ define

$$D^p f = \partial^{|p|} f / \partial y_1^{p_1} \dots \partial y_{2n}^{p_{2n}}.$$

Here $D^0 f = f$.

For the set $\mathcal{A}_k$ we consider the uniformity which is determined by the following base:

$$E_k(N, \epsilon, \lambda) = \{(f, g) \in \mathcal{A}_k \times \mathcal{A}_k : \ |D^p f(x, u) - D^p g(x, u)| \leq \epsilon$$

$$\text{for all } u, x \in R^n \text{ satisfying } |x|, |u| \leq N$$

$$\text{and each } p \in \{0, \dots, k\}^{2n} \text{ satisfying } |p| \leq k\}$$

$$\cap \{(f, g) \in \mathcal{A}_k \times \mathcal{A}_k : \ (|f(x, u)| + 1)(|g(x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for all } x, u \in R^n \text{ satisfying } |x| \leq N\},$$

where $N, \epsilon > 0$ and $\lambda > 1$. It is known (see Chapter 5 of [30]) that the uniform space $\mathcal{A}_k$ is metrizable and complete.

Set $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{M}_0 = \mathcal{M}$. For each integer $k \geq 1$ set $\mathcal{M}_k = \mathcal{M} \cap \mathcal{A}_k$.

Let $k \geq 0$ be an integer. Denote by $\bar{\mathcal{M}}_k$ the closure of $\mathcal{M}_k$ in $\mathcal{A}_k$ and consider the topological subspace $\bar{\mathcal{M}}_k \subset \mathcal{A}_k$ equipped with the relative topology.

Denote by $\mathcal{L}$ the set of all $f \in \mathcal{M} \cap C^2(R^{2n})$ such that

$$\partial f / \partial u_i \in C^2(R^{2n}) \text{ for } i = 1, \dots, n.$$

For any $k \in \{0, 1, 2\}$ denote by $\mathcal{L}_k$ the closure of $\mathcal{L}$ in the space $\mathcal{A}_k$ and consider the topological subspace $\mathcal{L}_k \subset \mathcal{A}_k$ equipped with the relative topology.

In [29] we established the following generic turnpike result which shows that most integrands possess the turnpike property described above (for its proof see also Theorem 5.1.2 of [30]).

Theorem 1.4. *Let $\mathfrak{M}$ be one of the following spaces:*

$$\mathcal{L}_k, \ k = 0, 1, 2, \ \bar{\mathcal{M}}_q, \ q \geq 0.$$

Then there exists a set $\mathcal{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that each $f \in \mathcal{F}$ has (ATP) and the following property.

For each $\epsilon, K > 0$ there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq K, \ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta$$

the inequality $|v(t)| \leq M$ holds for all $t \in [0, T]$ and

$$dist(H(f), \{v(t) : \ t \in [\tau, \tau + l]\}) \leq \epsilon \tag{1.11}$$

for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (1.11) holds for each $\tau \in [0, T - l_0]$ and if $d(v(T), H(f)) \leq \delta$, then (1.11) holds for each $\tau \in [l_0, T - l]$.

Note that in [29,30] the result stated above was proved in the case when $\mathfrak{M}$ in any of the spaces $\mathcal{M}_q$, $q \geq 0$. In the case when $\mathfrak{M}$ is in any of the spaces $\mathcal{L}_q$, $q = 0, 1, 2$ Theorem 1.4 is proved with the same proof.

Our paper is organized as follows. Turnpike properties for problems (P_2) and (P_3) are considered in Section 2. Section 3 contains preliminaries. In Section 4 we discuss the structure of approximate solutions of Lagrange problems (P_2) and (P_3) in the regions close to the endpoint of time intervals. In Section 5 we begin to study Bolza problems (P_4) and (P_5) and state a boundedness result (Theorem 5.1) and turnpike results (Theorems 5.3-5.6). Section 6 contains results on the the structure of approximate solutions of Bolza problems (P_4) and (P_5) in the regions close to the endpoints of time intervals (Theorems 6.2-6.9). Auxiliary results are collected in Section 7. Theorem 5.1 is proved in Section 8. The proofs of Theorems 5.3, 5.4 and 5.6 are given in Sections 9, 10 and 11 respectively. In Section 12 we prove auxiliary results for Theorems 6.2 and 6.3 which are proved in Sections 13 and 14 respectively. Section 15 contains auxiliary results for Theorems 6.5 and 6.8 which are proved in Section 16.

2. Turnpike results for problems (P_2) and (P_3)

Theorems 1.2-1.4 establish the turnpike property for the problems (P_1). In this section we obtain their analogs for the problems (P_2) and (P_3).

For $f \in \mathcal{A}$, $x \in R^n$ and a real number $T > 0$ set

$$U^f(T,x) = \inf\{I^f(0,T,v) : \ v : \ [0,T] \to R^n$$

$$\text{is an a.c. function satisfying } v(0) = x\}, \tag{2.1}$$

$$U^f(T) = \inf\{I^f(0,T,v) : \ v : \ [0,T] \to R^n \text{ is an a.c. function}\}. \tag{2.2}$$

The following result plays an important role in our study.

Theorem 2.1. *Let $f \in \mathcal{A}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0,\infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:*

(i) *if an a. c. function $v : [T_1, T_2] \to R^n$ satisfies*

$$|v(T_i)| \leq M_1, \ i = 1, 2, \ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + M_2,$$

then

$$|v(t)| \leq S, \ t \in [T_1, T_2]; \tag{2.3}$$

(ii) *if an a. c. function $v : [T_1, T_2] \to R^n$ satisfies*

$$|v(T_1)| \leq M_1, \ I^g(T_1, T_2, v) \leq U^g(T_2 - T_1, v(T_1)) + M_2,$$

then (2.3) holds;

(iii) *if an a. c. function $v : [T_1, T_2] \to R^n$ satisfies*

$$I^g(T_1, T_2, v) \leq U^g(T_2 - T_1) + M_2,$$

then (2.3) holds.

The properties (i) and (ii) were established in [28]. See also Theorem 1.2.3 of [30]. The property (iii) is proved analogously to the properties (i) and (ii).

Theorems 1.2 and 2.1 imply the following turnpike result.

Theorem 2.2. *Assume that an integrand $f \in \mathcal{A}$ has the asymptotic turnpike property and that $M_0, M_1, \epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, numbers $l, S > 0$ and integers $L, Q_* \geq 1$ such that for each $g \in \mathcal{U}$, each $T \geq L + lQ_*$ and each a.c. function $v : [0,T] \to R^n$ which satisfies at least one of the following conditions:*

$$|v(0)|, |v(T)| \leq M_1,\ I^g(0,T,v) \leq U^g(0,T,v(0),v(T)) + M_0;$$

$$|v(0)| \leq M_1,\ I^g(0,T,v) \leq U^g(T,v(0)) + M_0;$$

$$I^g(0,T,v) \leq U^g(T) + M_0$$

the inequality $|v(t)| \leq S$ holds for all $t \in [0,T]$ and there exist sequences of numbers $\{b_i\}_{i=1}^Q$, $\{c_i\}_{i=1}^Q \subset [0,T]$ such that

$$Q \leq Q_*,\ 0 \leq c_i - b_i \leq l,\ i = 1, \dots, Q$$

and that

$$dist(H(f), \{v(t) :\ t \in [\tau, \tau + L]\}) \leq \epsilon$$

for each $\tau \in [0, T-L] \setminus \cup_{i=1}^Q [b_i, c_i]$.

Theorems 1.3 and 2.1 imply the following result.

Theorem 2.3. *Assume that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property and that $\epsilon, K > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies at least one of the two conditions*

$$|v(0)| \leq K,\ I^g(0,T,v) \leq U^g(T,v(0)) + \delta;$$

$$I^g(0,T,v) \leq U^g(T) + \delta$$

the inequality $|v(t)| \leq M$ holds for all $t \in [0,T]$ and that

$$dist(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \leq \epsilon \tag{2.4}$$

for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (2.4) holds for each $\tau \in [0, T - l_0]$ and if $d(v(T), H(f)) \leq \delta$, then (2.4) holds for each $\tau \in [l_0, T - l]$.

Theorems 1.4 and 2.1 imply the following result.

Theorem 2.4. *Let $\mathfrak{M}$ be one of the following spaces:*

$$\mathcal{L}_k,\ k = 0,1,2,\ \bar{\mathcal{M}}_q,\ q \geq 0$$

and let the set $\mathcal{F} \subset \mathfrak{M}$ be as guaranteed by Theorem 1.4. Assume that $f \in \mathcal{F}$ and that $\epsilon, K > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies at least one of the two conditions

$$|v(0)| \leq K,\ I^g(0,T,v) \leq U^g(T,v(0)) + \delta;$$

$$I^g(0,T,v) \leq U^g(T) + \delta$$

the inequality $|v(t)| \leq M$ holds for all $t \in [0,T]$ and that

$$dist(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \leq \epsilon \tag{2.5}$$

for each $\tau \in [l_0, T - l_0]$. *Moreover, if* $d(v(0), H(f)) \leq \delta$, *then (2.5) holds for each* $\tau \in [0, T - l_0]$ *and if* $d(v(T), H(f)) \leq \delta$, *then (2.5) holds for each* $\tau \in [l_0, T - l]$.

3. Preliminaries

For each $f \in \mathcal{A}$ define

$$\bar{f}(x, y) = f(x, -y),\ x, y \in R^n. \tag{3.1}$$

It is clear that for each $f \in \mathcal{A}$, $\bar{f} \in \mathcal{A}$, if $f \in \mathcal{M}$, then $\bar{f} \in \mathcal{M}$, the mapping $f \to \bar{f}$, $f \in \mathcal{A}$ is continuous. This implies that $\bar{f} \in \bar{\mathcal{M}}$ for each $f \in \bar{\mathcal{M}}$. It is easy to see that for each integer $k \geq 1$, $\bar{f} \in \mathcal{A}_k$ for all $f \in \mathcal{A}_k$, for each integer $k \geq 0$, $\bar{f} \in \mathcal{M}_k$ for all $f \in \mathcal{M}_k$ and that the mapping $f \to \bar{f}$, $f \in \mathcal{A}_k$ is continuous. This implies that for each integer $k \geq 0$, $\bar{f} \in \bar{\mathcal{M}}_k$ for each $f \in \bar{\mathcal{M}}_k$. Evidently, $\bar{f} \in \mathcal{L}$ for all $f \in \mathcal{L}$ and for any $k \in \{0, 1, 2\}$ and any $f \in \mathcal{L}_k$, $\bar{f} \in \mathcal{L}_k$.

Let $f \in \mathcal{A}$. For any $T > 0$ and any a. c. function $v : [0, T] \to R^n$, put

$$\bar{v}(t) = v(T - t),\ t \in [0, T]. \tag{3.2}$$

Clearly, for each $T > 0$ and each a. c. function $v : [0, T] \to R^n$,

$$\int_0^T \bar{f}(\bar{v}(t), \bar{v}'(t))dt = \int_0^T f(v(T - t), v'(T - t))dt = \int_0^T f(v(t), v'(t))dt. \tag{3.3}$$

The next result easily follows from (3.3).

Proposition 3.1. *Let* $f \in \mathcal{A}$, $T > 0$, $M \geq 0$ *and* $v_i : [0, T] \to R^n$, $i = 1, 2$ *be an a. c. functions. Then*

$$I^f(0, T, v_1) \leq I^f(0, T, v_2) + M \textit{ if and only if } I^{\bar{f}}(0, T, \bar{v}_1) \leq I^{\bar{f}}(0, T, \bar{v}_2) + M.$$

For each $f \in \mathcal{A}$, each $x \in R^n$ and each real number $T > 0$ set

$$U_f(T, x) = \inf\{I^f(0, T, v) :\ v :\ [0, T] \to R^n$$

$$\text{is an a.c. function satisfying } v(T) = x\}, \tag{3.4}$$

Proposition 3.1 implies the following result.

Proposition 3.2. *Let* $f \in \mathcal{A}$, $T > 0, M \geq 0$ *and* $v : [0, T] \to R^n$ *be an a. c. function. Then*

$$\textit{if } I^f(0, T, v) \leq U^f(T) + M,\ \textit{then } I^{\bar{f}}(0, T, \bar{v}) \leq U^{\bar{f}}(T) + M;$$

$$\textit{if } I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + M,$$

$$\textit{then } I^{\bar{f}}(0, T, \bar{v}) \leq U^{\bar{f}}(0, T, \bar{v}(0), \bar{v}(T)) + M;$$

$$\textit{if } I^f(0, T, v) \leq U_f(T, v(T)) + M,\ \textit{then } I^{\bar{f}}(0, T, \bar{v}) \leq U^{\bar{f}}(T, \bar{v}(0)) + M;$$

$$\textit{if } I^f(0, T, v) \leq U^f(T, v(0)) + M,\ \textit{then } I^{\bar{f}}(0, T, \bar{v}) \leq U_{\bar{f}}(T, \bar{v}(T)) + M.$$

The next result follows from Proposition 3.2, Theorem 2.1 and the continuity of the mapping $f \to \bar{f}$, $f \in \mathcal{A}$.

Proposition 3.3. *Let $f \in \mathcal{A}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and $S > 0$ such that for each $g \in \mathcal{U}$, each $T \geq c$ and each a. c. function $v : [0, T] \to R^n$ which satisfies*

$$I^g(0, T, v) \leq U_g(T, v(T)) + M_2, \ |v(T)| \leq M_1$$

the inequality $|v(t)| \leq S$ holds for all $t \in [0, T]$.

The following result was proved in [36].

Proposition 3.4. *Assume that $f \in \mathcal{A}$ has (ATP). Then $\bar{f}$ has (ATP) and $H(\bar{f}) = H(f)$.*

The following result is proved in [30] (see Chapter 4, Proposition 4.2.1).

Proposition 3.5. *Let $f \in \mathcal{A}$. Then $\pi^f(x) \to \infty$ as $|x| \to \infty$.*

Let $f \in \mathcal{A}$. Define

$$\mathcal{D}(f) = \{x \in R^n : \ \pi^f(x) \leq \pi^f(y) \text{ for all } y \in R^n\}. \tag{3.5}$$

Since the function π^f is continuous it follows from Proposition 3.5 that the set $\mathcal{D}(f)$ is nonempty, bounded and closed.

For each $\tau_1 \in R^1$, $\tau_2 > \tau_1$, each $r_1, r_2 \in [\tau_1, \tau_2]$ satisfying $r_1 < r_2$ and each a.c. function $u : [\tau_1, \tau_2] \to R^n$ set

$$\Gamma^f(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1)\mu(f). \tag{3.6}$$

In view of (1.2), (1.5), (1.6) and (3.6),

$$\Gamma^f(r_1, r_2, u) \geq 0 \text{ for each } \tau_1 \in R^1, \ \tau_2 > \tau_1, \ \text{ each } r_1, r_2 \in [\tau_1, \tau_2]$$

$$\text{satisfying } r_1 < r_2 \text{ and each a.c. function } u : [\tau_1, \tau_2] \to R^n. \tag{3.7}$$

Proposition 3.6 (Theorem 3.6.3 of [30])**.** *Let $f \in \mathcal{A}$. For every $x \in R^n$ there exists an (f)-good function $v : [0, \infty) \to R^n$ such that $v(0) = x$ and $\Gamma^f(T_1, T_2, v) = 0$ for each $T_1 \geq 0$ and each $T_2 > T_1$.*

Let $f \in \mathcal{A}$. An a. c. function $v : [0, \infty) \to R^n$ is called (f)-perfect if $\Gamma^f(T_1, T_2, v) = 0$ for all $T_1 \geq 0$ and all $T_2 > T_1$.

Propositions 3.5 and 1.1 imply the following result.

Proposition 3.7. *Let $f \in \mathcal{A}$ and $v : [0, \infty) \to R^n$ be an (f)-perfect function. Then the function v is bounded and (f)-good.*

4. Structure of solutions of Lagrange problems near the endpoints

For each $f \in \mathcal{A}$ and each $x \in R^n$ denote by $\mathcal{P}(f, x)$ the set of all (f)-perfect functions $v : [0, \infty) \to R^n$ such that $v(0) = x$. In view of Proposition 3.6 this set is nonempty.

Let $f \in \mathcal{A}$ and $x \in R^n$. By Proposition 3.7 any function belonging to $\mathcal{P}(f, x)$ is bounded and (f)-good. The following results were obtained in [36].

Proposition 4.1. *Let $f \in \mathcal{A}$ and D be a nonempty bounded subset of R^n. Then there exist a number $S > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $x \in D$, each $g \in \mathcal{U}$ and each $v \in \mathcal{P}(g, x)$, $|v(t)| \leq S$ for all $t \geq 0$.*

Theorem 4.2. *Suppose that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property. Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $\delta \in (0,\epsilon)$ and $T_0 > \tau_0$ such that for each $g \in \mathcal{U}$, each $T \geq T_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$|v(0)| \leq M,\ I^g(0,T,v) \leq U^g(T, v(0)) + \delta$$

there exists an a. c. function $w \in \cup\{\mathcal{P}(\bar{f}, z) :\ z \in \mathcal{D}(\bar{f})\}$ such that $|v(T-t) - w(t)| \leq \epsilon$ for all $t \in [0,\tau_0]$.

Theorem 4.3. *Suppose that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property. Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $\delta \in (0,\epsilon)$ and $T_0 > \tau_0$ such that for each $g \in \mathcal{U}$, each $T \geq T_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$|v(T)| \leq M,\ I^g(0,T,v) \leq U_g(T, v(T)) + \delta$$

there exists an a. c. function $w \in \cup\{\mathcal{P}(f, z) :\ z \in \mathcal{D}(f)\}$ such that $|v(t) - w(t)| \leq \epsilon$ for all $t \in [0,\tau_0]$.

Theorem 4.4. *Suppose that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property. Let $\epsilon, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $\delta \in (0,\epsilon)$ and $T_0 > \tau_0$ such that for each $g \in \mathcal{U}$, each $T \geq T_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$I^g(0,T,v) \leq U^g(T) + \delta$$

there exist a. c. functions

$$w_1 \in \cup\{\mathcal{P}(f,z) :\ z \in \mathcal{D}(f)\} \text{ and } w_2 \in \cup\{\mathcal{P}(\bar{f},z) :\ z \in \mathcal{D}(\bar{f})\}$$

such that $|v(t) - w_1(t)| \leq \epsilon$ and $|v(T-t) - w_2(t)| \leq \epsilon$ for all $t \in [0,\tau_0]$.

Theorem 4.5. *Let $\mathfrak{M}$ be one of the following spaces:*

$$\mathcal{L}_k,\ k = 0,1,2,\ \bar{\mathcal{M}}_q,\ q \geq 3.$$

Then there exists a set $\mathcal{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that for each $f \in \mathcal{F}$ there exist a unique pair of points $z_f, z_{\bar{f}} \in R^n$ satisfying $\mathcal{D}(f) = \{z_f\}$ and $\mathcal{D}(\bar{f}) = \{z_{\bar{f}}\}$, a unique (f)-perfect function $v_f : [0,\infty) \to R^n$ satisfying $v_f(0) = z_f$ and a unique $(\bar{f})$-perfect function $v_{\bar{f}} : [0,\infty) \to R^n$ satisfying $v_{\bar{f}}(0) = z_{\bar{f}}$ and such that the following assertion holds.

Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $\delta \in (0,\epsilon)$ and $T_0 > \tau_0$ such that for each $g \in \mathcal{U}$, each $T \geq T_0$ and each a.c. function $v : [0,T] \to R^n$,

$$\text{if } |v(0)| \leq M,\ I^g(0,T,v) \leq U^g(T, v(0)) + \delta, \text{ then}$$
$$|v(T-t) - v_{\bar{f}}(t)| \leq \epsilon \text{ for all } t \in [0,\tau_0],$$
$$\text{if } |v(T)| \leq M,\ I^g(0,T,v) \leq U_g(T, v(T)) + \delta, \text{ then}$$
$$|v(t) - v_f(t)| \leq \epsilon \text{ for all } t \in [0,\tau_0],$$
$$\text{if } I^g(0,T,v) \leq U^g(T) + \delta, \text{ then}$$
$$|v(T-t) - v_{\bar{f}}(t)| \leq \epsilon \text{ and } |v(t) - v_f(t)| \leq \epsilon \text{ for all } t \in [0,\tau_0].$$

5. Turnpike properties of Bolza problems

Denote my mes(E) the Lebesgue measure of a Lebesgue measurable set $E \subset R^1$. Let $a_1 > 0$. Denote by $\mathfrak{A}(R^n)$ the set of all lower semicontinuous functions $h : R^n \to R^1$ which are bounded on bounded subsets of R^n and satisfy

$$h(z) \geq -a_1 \text{ for all } z \in R^n. \tag{5.1}$$

For simplicity we set $\mathfrak{A} = \mathfrak{A}(R^n)$. We equip the set $\mathfrak{A}$ with the uniformity which is determined by the following base:

$$E(N,\epsilon) = \{(h_1,h_2) \in \mathfrak{A} \times \mathfrak{A} : \ |h_1(z) - h_2(z)| \leq \epsilon$$

$$\text{for each } z \in R^n \text{ satisfying } |z| \leq N\}, \tag{5.2}$$

where $N, \epsilon > 0$. It is not difficult to see that the uniform space $\mathfrak{A}$ is metrizable and complete. We consider the following Bolza variational problems

$$I^g(T_1,T_2,v) + h(v(T_2)) \to \min, \ v(T_1) = y, \tag{P_4}$$

$$v : \ [T_1,T_2] \to R^n \text{ is an a. c. function,}$$

$$I^g(T_1,T_2,v) + h(v(T_2)) + \xi(v(T_1)) \to \min, \tag{P_5}$$

$$v : \ [T_1,T_2] \to R^n \text{ is an a. c. function,}$$

where $g \in \mathcal{A}$, $h, \xi \in \mathfrak{A}$, $y \in R^n$ and $-\infty < T_1 < T_2 < \infty$. Set

$$\sigma(g,h,y,T_1,T_2) = \inf\{I^g(T_1,T_2,v) + h(v(T_2)) :$$

$$v : \ [T_1,T_2] \to R^n \text{ is an a. c. function, } v(T_1) = y\}, \tag{5.3}$$

$$\sigma(g,h,\xi,T_1,T_2) = \inf\{I^g(T_1,T_2,v) + h(v(T_2)) + \xi(v(T_1)) :$$

$$v : \ [T_1,T_2] \to R^n \text{ is an a. c. function}\}, \tag{5.4}$$

$$\widehat{\sigma}(g,\xi,z,T_1,T_2) = \inf\{I^g(T_1,T_2,v) + \xi(v(T_1)) :$$

$$v : \ [T_1,T_2] \to R^n \text{ is an a. c. function, } v(T_2) = z\}, \tag{5.5}$$

$$\sigma(g,h,\xi,y,z,T_1,T_2) = \inf\{I^g(T_1,T_2,v) + h(v(T_2)) + \xi(v(T_1)) :$$

$$v : \ [T_1,T_2] \to R^n \text{ is an a. c. function, } v(T_1) = y, \ v(T_2) = z\}. \tag{5.6}$$

We begin with the following uniform boundedness result which is proved in Section 8.

Theorem 5.1. *Let $f \in \mathcal{A}$, $h_1, h_2 \in \mathfrak{A}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, a neighborhood $\mathcal{V}_i$ of h_i in $\mathfrak{A}$, $i = 1,2$ and $S > 0$ such that for each $g \in \mathcal{U}$, each $\xi_i \in \mathcal{V}_i$, $i = 1,2$ each $T_1 \in [0,\infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:*

(i) *if an a. c. function $v : [T_1,T_2] \to R^n$ satisfies*

$$|v(T_1)| \leq M_1,$$

$$I^g(T_1,T_2,v) + \xi_1(v(T_2)) \leq \sigma(g,\xi_1,v(T_1),T_1,T_2) + M_2,$$

then

$$|v(t)| \leq S, \ t \in [T_1,T_2]; \tag{5.7}$$

(ii) *if an a. c. function* $v : [T_1, T_2] \to R^n$ *satisfies*

$$I^g(T_1, T_2, v) + \xi_1(v(T_2)) + \xi_2(v(T_1)) \le \sigma(g, \xi_1, \xi_2, T_1, T_2) + M_2$$

then (5.7) holds.

Relation (3.3) implies the following result.

Proposition 5.2. *Let* $g \in \mathcal{A}$, $h \in \mathfrak{A}$, $T > 0, M \ge 0$, $v : [0, T] \to R^n$ *be an a. c. function and* $\bar{v}(t) = v(T - t)$, $t \in [0, T]$. *Then the following assertions hold:*

$$I^g(0, T, v) + h(v(T)) + \xi(v(0)) \le \sigma(g, h, \xi, 0, T) + M$$

if and only if $I^{\bar{g}}(0, T, \bar{v}) + \xi(\bar{v}(T)) + h(\bar{v}(0)) \le \sigma(\bar{g}, \xi, h, 0, T) + M$;

$$I^g(0, T, v) + h(v(T)) \le \sigma(g, h, v(0), 0, T) + M$$

if and only if $I^{\bar{g}}(0, T, \bar{v}) + h(\bar{v}(0)) \le \widehat{\sigma}(\bar{g}, h, \bar{v}(T), 0, T) + M$;

$$I^g(0, T, v) + h(v(0)) \le \widehat{\sigma}(g, h, v(T), 0, T) + M$$

if and only if $I^{\bar{g}}(0, T, \bar{v}) + h(\bar{v}(T)) \le \sigma(\bar{g}, h, \bar{v}(0), 0, T) + M$.

Let $f \in \mathcal{A}$ have (ATP). By Theorem 2.1, there exist a neighborhood $\mathcal{U}_f$ of f in $\mathcal{A}$ and $S_f > 0$ such that the following properties hold:

(P1) for each $g \in \mathcal{U}_f$, each $T \ge 1$ and each a. c. function $u : [0, T] \to R^n$ satisfying

$$I^g(0, T, u) \le U^g(T) + 1$$

we have

$$|u(t)| \le S_f, \ t \in [0, T];$$

(P2) for each $g \in \mathcal{U}_f$, each $T \ge 1$ and each a. c. function $u : [0, T] \to R^n$ satisfying

$$d(u(0), H(f)) \le 1,$$
$$I^g(0, T, u) \le U^g(T, u(0)) + 1$$

we have

$$|u(t)| \le S_f, \ t \in [0, T].$$

The following turnpike results for Bolza variational problems show that the turnpike phenomenon, for approximate solutions on large intervals, is stable under small perturbations of the objective functions.

Theorem 5.3. *Assume that an integrand* $f \in \mathcal{A}$ *has the asymptotic turnpike property and that* $M_0, M_1, M_2, \epsilon > 0$. *Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$, *numbers* $l, S > 0$ *and integers* $L, Q_* \ge 1$ *such that for each* $g \in \mathcal{U}$, *each* $T \ge L + lQ_*$, *each* $h, \xi \in \mathfrak{A}$ *satisfying*

$$|h(z)|, \ |\xi(z)| \le M_2 \text{ for all } z \in R^n \text{ such that } |z| \le S_f$$

and each a.c. function $v : [0, T] \to R^n$ *which satisfies at least one of the following conditions:*

(a)

$$|v(0)| \le M_0,$$
$$I^g(0, T, v) + h(v(T)) \le \sigma(g, h, v(0), 0, T) + M_1;$$

(b)

$$I^g(0,T,v)+h(v(T))+\xi(v(0))\le\sigma(g,h,\xi,0,T)+M_1$$

the inequality $|v(t)|\le S$ *holds for all* $t\in[0,T]$ *and there exist sequences of numbers* $\{b_i\}_{i=1}^q$, $\{c_i\}_{i=1}^q\subset[0,T]$ *such that*

$$q\le Q_*,\ 0\le c_i-b_i\le l,\ i=1,\dots,q$$

and that

$$dist(H(f),\{v(t):\ t\in[\tau,\tau+L]\})\le\epsilon$$

for each $\tau\in[0,T-L]\setminus\cup_{i=1}^q[b_i,c_i]$.

Theorem 5.3 is proved in Section 9.

The next result is proved in Section 10.

Theorem 5.4. *Assume that an integrand* $f\in\mathcal{M}$ *has the asymptotic turnpike property and that* $\epsilon,M_0>0$. *Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$ *and numbers* $M_1>M_0$, $l_1>l>0$, $\delta>0$ *such that for each* $g\in\mathcal{U}$, *each* $T\ge 2l_1+l$ *and each a.c. function* $v:[0,T]\to R^n$ *which satisfies*

$$|v(0)|,\ |v(T)|\le M_0,$$

$$I^g(0,T,v)\le U^g(0,T,v(0),v(T)))+M_0$$

and

$$I^g(S,S+l_1,v)\le U^g(S,S+l_1,v(S),v(S+l_1))+\delta$$

for each $S\in[0,T-l_1]$, *the inequality* $|v(t)|\le M_1$ *holds for all* $t\in[0,T]$ *and that there exist*

$$\tau_1\in[0,l_1],\ \tau_2\in[T-l_1,T]$$

such that for all $\tau\in[\tau_1,\tau_2-l]$,

$$dist(H(f),\{v(t):\ t\in[\tau,\tau+l]\})\le\epsilon.$$

Moreover, if $d(v(0),H(f))\le\delta$, *then* $\tau_1=0$ *nd if* $d(v(T),H(f))\le\delta$, *then* $\tau_2=T$.

Theorems 2.1 and 5.4 imply the following result.

Theorem 5.5. *Assume that an integrand* $f\in\mathcal{M}$ *has the asymptotic turnpike property and that* $\epsilon,M_0,M_1>0$. *Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$ *and numbers* $M_2>M_1,M_0$, $l_1>l>0$, $\delta>0$ *such that for each* $g\in\mathcal{U}$, *each* $T\ge 2l_1+l$ *and each a.c. function* $v:[0,T]\to R^n$ *which satisfies at least one of the following conditions below:*

$$|v(0)|,\ |v(T)|\le M_0,$$

$$I^g(0,T,v)\le U^g(0,T,v(0),v(T))+M_1;$$

$$|v(0)|\le M_0,\ I^g(0,T,v)\le U^g(T,v(0))+M_1;$$

$$I^g(0,T,v)\le U^g(T)+M_1$$

and

$$I^g(S,S+l_1,v)\le U^g(S,S+l_1,v(S),v(S+l_1))+\delta$$

for each $S\in[0,T-l_1]$, *the inequality* $|v(t)|\le M_2$ *holds for all* $t\in[0,T]$ *and that there exist*

$$\tau_1\in[0,l_1],\ \tau_2\in[T-l_1,T]$$

such that for all $\tau \in [\tau_1, \tau_2 - l]$,

$$dist(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon.$$

Moreover, if $d(v(0), H(f)) \leq \delta$, *then* $\tau_1 = 0$ *and if* $d(v(T), H(f)) \leq \delta$, *then* $\tau_2 = T$.

The next result is proved in Section 11.

Theorem 5.6. *Assume that an integrand* $f \in \mathcal{M}$ *has the asymptotic turnpike property and that* $\epsilon, M_0, M_1, M_2 > 0$. *Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$ *and numbers* $S > 0$, $L > l > 0$, $\delta > 0$ *such that for each* $g \in \mathcal{U}$, *each* $h, \xi \in \mathfrak{A}$ *satisfying*

$$|h(z)|, \ |\xi(z)| \leq M_2 \text{ for all } z \in R^n \text{ such that } |z| \leq S_f,$$

each $T \geq 2L + l$ *and each a.c. function* $v : [0, T] \to R^n$ *which satisfies at least one of the following conditions below:*

(a) $|v(0)| \leq M_0, \ I^g(0, T, v) + h(v(T)) \leq \sigma(g, h, v(0), 0, T) + M_1$;

(b)

$$I^g(0, T, v) + h(v(T)) + \xi(v(0)) \leq \sigma(g, h, \xi, 0, T) + M_1$$

and such that for each $\tau \in [0, T - L]$,

$$I^g(\tau, \tau + L, v) \leq U^g(\tau, \tau + L, v(\tau), v(\tau + L)) + \delta$$

the inequality $|v(t)| \leq S$ *holds for all* $t \in [0, T]$ *and that there exist*

$$\tau_1 \in [0, L], \ \tau_2 \in [T - L, T]$$

such that for all $\tau \in [\tau_1, \tau_2 - l]$,

$$dist(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon.$$

Moreover, if $d(v(0), H(f)) \leq \delta$, *then* $\tau_1 = 0$ *and if* $d(v(T), H(f)) \leq \delta$, *then* $\tau_2 = T$.

6. Structure of solutions of Bolza problems near the endpoints

For each nonempty set X and each $\eta : X \to R^1$ define

$$\inf(\eta) = \inf\{\eta(x) : x \in X\}.$$

Let $f \in \mathcal{M}$ have the asymptotic turnpike property and let $h, \xi \in \mathfrak{A}$. Proposition 3.5 and (5.1) imply the following result.

Proposition 6.1. *The function* $\pi^f + h$ *is lower semicontinuous, for every* $M > 0$ *the set*

$$\{x \in R^n : (\pi^f + h)(x) \leq M\}$$

is bounded, $\inf(\pi^f + h)$ *is finite and the function* $\pi^f + h$ *has a point of minimum.*

The next result is proved in Section 13.

Theorem 6.2. *Suppose that an integrand* $f \in \mathcal{M}$ *has the asymptotic turnpike property and that* $h \in \mathfrak{A}$. *Let* $\epsilon, M, L_0 > 0$. *Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$, *a neighborhood* $\mathcal{V}$ *of* h *in* $\mathfrak{A}$ *and numbers* $\delta \in (0, \epsilon)$ *and* $T_0 > L_0$ *such that for each* $T \geq T_0$, *each* $g \in \mathcal{U}$, *each* $\xi \in \mathcal{V}$ *and each a.c. function* $v : [0, T] \to R^n$ *which satisfies*

$$|v(0)| \leq M,$$

$$I^g(0, T, v) + \xi(v(T)) \leq \sigma(g, \xi, v(0), 0, T) + \delta$$

there exists an $(\bar{f})$*-perfect function* $w : [0, \infty) \to R^n$ *such that*

$$(\pi^{\bar{f}} + h)(w(0)) = \inf(\pi^{\bar{f}} + h),$$

$$|v(T - t) - w(t)| \le \epsilon \text{ for all } t \in [0, L_0].$$

The proof of the following result is given in Section 14.

Theorem 6.3. *Suppose that an integrand* $f \in \mathcal{M}$ *has the asymptotic turnpike property and that* $h_1, h_2 \in \mathfrak{A}$*. Let* $\epsilon, L_0 > 0$*. Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$*, a neighborhood* $\mathcal{V}_i$ *of* h_i*,* $i = 1, 2$ *in* $\mathfrak{A}$ *and numbers* $\delta \in (0, \epsilon)$ *and* $T_0 > L_0$ *such that for each* $T \ge T_0$*, each* $g \in \mathcal{U}$*, each* $\xi_i \in \mathcal{V}_i$*,* $i = 1, 2$ *and each a.c. function* $v : [0, T] \to R^n$ *which satisfies*

$$I^g(0, T, v) + \xi_1(v(T)) + \xi_2(v(0)) \le \sigma(g, \xi_1, \xi_2, 0, T) + \delta$$

there exists an (f)*-perfect function* $w_1 : [0, \infty) \to R^n$ *and an* $(\bar{f})$*-perfect function* $w_2 : [0, \infty) \to R^n$*such that*

$$(\pi^f + h_2)(w_1(0)) = \inf(\pi^f + h_2),$$

$$(\pi^{\bar{f}} + h_1)(w_2(0)) = \inf(\pi^{\bar{f}} + h_1),$$

$$|v(t) - w_1(t)| \le \epsilon \text{ for all } t \in [0, L_0].$$

$$|v(T - t) - w_2(t)| \le \epsilon \text{ for all } t \in [0, L_0].$$

Theorem 6.4. *Let* $h \in \mathfrak{A}$ *and* $\mathfrak{M}$ *be one of the following spaces:*

$$\mathcal{L}_k,\ k = 0, 1, 2,\ \bar{\mathcal{M}}_q,\ q \ge 3.$$

Then there exists a set $\mathcal{F} \subset \mathfrak{M}$ *which is a countable intersection of open everywhere dense subsets of* $\mathfrak{M}$ *such that for each* $f \in \mathcal{F}$ *there exist a unique pair of points* $z_f, z_{\bar{f}} \in R^n$ *satisfying*

$$\{z \in R^n :\ (\pi^f + h)(z) = \inf(\pi^f + h)\} = \{z_f\},$$

$$\{z \in R^n :\ (\pi^{\bar{f}} + h)(z) = \inf(\pi^{\bar{f}} + h)\} = \{z_{\bar{f}}\},$$

a unique (f)*-perfect function* $v_f : [0, \infty) \to R^n$ *satisfying* $v_f(0) = z_f$ *and a unique* $(\bar{f})$*-perfect function* $v_{\bar{f}} : [0, \infty) \to R^n$ *satisfying* $v_{\bar{f}}(0) = z_{\bar{f}}$ *and such that the following assertion holds.*

Let $\epsilon, M, \tau_0 > 0$*. Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$*, a neighborhood* $\mathcal{V}$ *of* h *in* $\mathfrak{A}$ *and numbers* $\delta \in (0, \epsilon)$ *and* $T_0 > \tau_0$ *such that for each* $g \in \mathcal{U}$*, each* $\xi \in \mathcal{V}$*, each* $T \ge T_0$ *and each a.c. function* $v : [0, T] \to R^n$*,*

$$\text{if } |v(0)| \le M,\ I^g(0, T, v) + \xi(v(T)) \le \sigma(g, \xi, v(0), 0, T) + \delta, \text{ then}$$

$$|v(T - t) - v_{\bar{f}}(t)| \le \epsilon \text{ for all } t \in [0, \tau_0];$$

$$\text{if } |v(T)| \le M,\ I^g(0, T, v) + \xi(v(0)) \le \widehat{\sigma}(g, \xi, v(T), 0, T) + \delta, \text{ then}$$

$$|v(t) - v_f(t)| \le \epsilon \text{ for all } t \in [0, \tau_0].$$

Theorem 6.4 follows from the continuity of the mapping $f \to \bar{f}$, $f \in \mathcal{A}_k$, $k = 0, 1, \dots,$ Proposition 5.2 and the following result which is proved in Section 16.

Theorem 6.5. *Let $h \in \mathfrak{A}$ and $\mathfrak{M}$ be one of the following spaces:*

$$\mathcal{L}_k,\ k = 0, 1, 2,\ \bar{\mathcal{M}}_q,\ q \geq 3.$$

Then there exists a set $\mathcal{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that for each $f \in \mathcal{F}$ there exist a unique point $z_f \in R^n$ satisfying

$$\{z \in R^n :\ (\pi^f + h)(z) = \inf(\pi^f + h)\} = \{z_f\}$$

and a unique (f)-perfect function $v_f : [0, \infty) \to R^n$ satisfying $v_f(0) = z_f$ and such that the following assertion holds.

Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, a neighborhood $\mathcal{V}$ of h in $\mathfrak{A}$ and numbers $\delta \in (0, \epsilon)$ and $T_0 > \tau_0$ such that for each $g \in \mathcal{U}$, each $\xi \in \mathcal{V}$, each $T \geq T_0$ and each a.c. function $v : [0, T] \to R^n$ such that

$$|v(T)| \leq M,\ I^g(0, T, v) + \xi(v(0)) \leq \widehat{\sigma}(g, \xi, v(T), 0, T) + \delta$$

the inequality $|v(t) - v_f(t)| \leq \epsilon$ holds for all $t \in [0, \tau_0]$.

Theorems 5.1 and 6.4 imply the following result.

Theorem 6.6. *Let $h_1, h_2 \in \mathfrak{A}$ and $\mathfrak{M}$ be one of the following spaces:*

$$\mathcal{L}_k,\ k = 0, 1, 2,\ \bar{\mathcal{M}}_q,\ q \geq 3.$$

Then there exists a set $\mathcal{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that for each $f \in \mathcal{F}$ there exist a unique pair of points $z_f, z_{\bar{f}} \in R^n$ satisfying

$$\{z \in R^n :\ (\pi^f + h_1)(z) = \inf(\pi^f + h_1)\} = \{z_f\},$$

$$\{z \in R^n :\ (\pi^{\bar{f}} + h_2)(z) = \inf(\pi^{\bar{f}} + h_2)\} = \{z_{\bar{f}}\},$$

a unique (f)-perfect function $v_f : [0, \infty) \to R^n$ satisfying $v_f(0) = z_f$ and a unique $(\bar{f})$-perfect function $v_{\bar{f}} : [0, \infty) \to R^n$ satisfying $v_{\bar{f}}(0) = z_{\bar{f}}$ and such that the following assertion holds.

Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, a neighborhood $\mathcal{V}_i$ of h_i, $i = 1, 2$ in $\mathfrak{A}$ and numbers $\delta \in (0, \epsilon)$ and $T_0 \geq \tau_0$ such that for each $g \in \mathcal{U}$, each $\xi_i \in \mathcal{V}_i$, $i = 1, 2$, each $T \geq T_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$I^g(0, T, v) + \xi_1(v(T)) + \xi_2(v(0)) \leq \sigma(g, \xi_1, \xi_2, , 0, T) + \delta,$$

for all $t \in [0, \tau_0]$, $|v(T - t) - v_{\bar{f}}(t)| \leq \epsilon$, $|v(t) - v_f(t)| \leq \epsilon$.

In the next theorems $\mathfrak{M}$ is one of the following spaces:

$$\mathcal{L}_k,\ k = 0, 1, 2,\ \bar{\mathcal{M}}_q,\ q \geq 3$$

and the spaces $\mathfrak{M} \times \mathfrak{A}$ and $\mathfrak{M} \times \mathfrak{A} \times \mathfrak{A}$ are equipped with the product topology.

Theorem 6.7. *There exists a set $\mathcal{F} \subset \mathfrak{M} \times \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M} \times \mathfrak{A}$ such that for each $(f, h) \in \mathcal{F}$ there exist a unique pair of points $z_{f,h}, z_{\bar{f},h} \in R^n$ satisfying*

$$\{z \in R^n :\ (\pi^f + h)(z) = \inf(\pi^f + h)\} = \{z_{f,h}\},$$

$$\{z \in R^n :\ (\pi^{\bar{f}} + h)(z) = \inf(\pi^{\bar{f}} + h)\} = \{z_{\bar{f},h}\},$$

a unique (f)-perfect function $v_{f,h} : [0,\infty) \to R^n$ satisfying $v_{f,h}(0) = z_{f,h}$ and a unique $(\bar{f})$-perfect function $v_{\bar{f},h} : [0,\infty) \to R^n$ satisfying $v_{\bar{f},h}(0) = z_{\bar{f},h}$ and such that the following assertion holds.

Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of (f,h) in $\mathfrak{M} \times \mathfrak{A}$ and numbers $\delta \in (0,\epsilon)$ and $T_0 > \tau_0$ such that for each $(g,\xi) \in \mathcal{U}$, each $T \geq T_0$ and each a.c. function $v : [0,T] \to R^n$,

$$\text{if } |v(0)| \leq M,\ I^g(0,T,v) + \xi(v(T)) \leq \sigma(g,\xi,v(0),0,T) + \delta, \text{ then}$$
$$|v(T-t) - v_{\bar{f},h}(t)| \leq \epsilon \text{ for all } t \in [0,\tau_0];$$
$$\text{if } |v(T)| \leq M,\ I^g(0,T,v) + \xi(v(0)) \leq \widehat{\sigma}(g,\xi,v(T),0,T) + \delta, \text{ then}$$
$$|v(t) - v_{f,h}(t)| \leq \epsilon \text{ for all } t \in [0,\tau_0].$$

Theorem 6.7 follows from the continuity of the mapping $f \to \bar{f}$, $f \in \mathcal{A}_k$, $k = 0,1,\ldots$, Proposition 5.2 and the following result which is proved in Section 16.

Theorem 6.8. *There exists a set $\mathcal{F} \subset \mathfrak{M} \times \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M} \times \mathfrak{A}$ such that for each $(f,h) \in \mathcal{F}$ there exist a unique point $z_{f,h} \in R^n$ satisfying*

$$\{z \in R^n :\ (\pi^f + h)(z) = \inf(\pi^f + h)\} = \{z_{f,h}\}$$

and a unique (f)-perfect function $v_{f,h} : [0,\infty) \to R^n$ satisfying $v_{f,h}(0) = z_{f,h}$ and such that the following assertion holds.

Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of (f,h) in $\mathfrak{M} \times \mathfrak{A}$ and numbers $\delta \in (0,\epsilon)$ and $T_0 > \tau_0$ such that for each $(g,\xi) \in \mathcal{U}$, each $T \geq T_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$|v(T)| \leq M,\ I^g(0,T,v) + \xi(v(0)) \leq \widehat{\sigma}(g,\xi,v(T),0,T) + \delta$$

the inequality $|v(t) - v_{f,h}(t)| \leq \epsilon$ holds for all $t \in [0,\tau_0]$.

Theorems 5.1 and 6.8 imply the following result.

Theorem 6.9. *Let the set $\mathcal{F} \subset \mathfrak{M} \times \mathfrak{A}$ be as guaranteed by Theorem 6.7 and let*

$$\mathcal{G} = \{(f,h_1,h_2) \in \mathfrak{M} \times \mathfrak{A} \times \mathfrak{A} :$$
$$(f,h_i) \in \mathcal{F},\ i = 1,2\}.$$

Then $\mathcal{G}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{M} \times \mathfrak{A} \times \mathfrak{A}$ such that for each $(f,h_1,h_2) \in \mathcal{G}$ there exist a unique pair of points $z_{*,1}, z_{*,2} \in R^n$ satisfying

$$\{z \in R^n :\ (\pi^f + h_2)(z) = \inf(\pi^f + h_2)\} = \{z_{*,1}\},$$
$$\{z \in R^n :\ (\pi^{\bar{f}} + h_1)(z) = \inf(\pi^{\bar{f}} + h_1)\} = \{z_{*,2}\},$$

a unique (f)-perfect function $v_1 : [0,\infty) \to R^n$ satisfying $v_1(0) = z_{*,1}$ and a unique $(\bar{f})$-perfect function $v_2 : [0,\infty) \to R^n$ satisfying $v_2(0) = z_{*,2}$ and such that the following assertion holds.

Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of (f,h_1,h_2) in $\mathfrak{M} \times \mathfrak{A} \times \mathfrak{A}$ and numbers $\delta \in (0,\epsilon)$ and $T_0 > \tau_0$ such that for each $(g,\xi_1,\xi_2) \in \mathcal{U}$, each $T \geq T_0$ and each a.c. function $v : [0,T] \to R^n$ satisfying

$$I^g(0,T,v) + \xi_1(v(T)) + \xi_2(v(0)) \leq \sigma(g,\xi_1,\xi_2,0,T) + \delta$$

for all $t \in [0,\tau_0]$, $|v(t) - v_1(t)| \leq \epsilon$, $|v(T-t) - v_2(t)| \leq \epsilon$.

7. Auxiliary results

Lemma 7.1 (Lemma 4.2.8 of [30]). *Let $f \in \mathcal{A}$ possess (ATP). Then*

$$\sup\{\pi^f(z) : z \in H(f)\} = 0.$$

Proposition 7.2 (Theorem 4.1.1 of [30]). *Assume that $f \in \mathcal{A}$ has (ATP). Then f is a continuity point of the mapping $g \to (\mu(g), \pi^g) \in R^1 \times C(R^n)$, $g \in \mathcal{A}$, where $C(R^n)$ is the space of all continuous functions $\phi : R^n \to R^1$ with the topology of the uniform convergence on bounded sets.*

Proposition 7.3 (Theorem 1.2.2 of [30]). *For each $f \in \mathcal{A}$ there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and a number $M > 0$ such that for each $g \in \mathcal{U}$ and each (g)-good function $x : [0, \infty) \to R^n$ the relation $\limsup_{t\to\infty} |x(t)| < M$ holds.*

Proposition 7.4 (Proposition 1.3.5 of [30]). *Assume that $f \in \mathcal{A}$, $M_1 > 0$, $0 \le T_1 < T_2 < \infty$ and that $x_i : [T_1, T_2] \to R^n$, $i = 1, 2, \dots$ is a sequence of a.c. functions such that $I^f(T_1, T_2, x_i) \le M_1$ for all integers $i \ge 1$. Then there exist a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ and an a.c. function $x : [T_1, T_2] \to R^n$ such that $I^f(T_1, T_2, x) \le M_1$, $x_{i_k}(t) \to x(t)$ as $k \to \infty$ uniformly on $[T_1, T_2]$ and $x'_{i_k} \to x'$ as $k \to \infty$ weakly in $L^1(R^n; (T_1, T_2))$.*

Corollary 7.5 (Corollary 1.3.1 of [30]). *For each $f \in \mathcal{A}$, each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ and each $z_1, z_2 \in R^n$ there is an a.c. function $x : [T_1, T_2] \to R^n$ such that $x(T_i) = z_i$, $i = 1, 2$ and $I^f(T_1, T_2, x) = U^f(T_1, T_2, z_1, z_2)$.*

Corollary 7.6. *For each $f \in \mathcal{A}$, each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ and each $z \in R^n$ there is an a.c. function $x : [T_1, T_2] \to R^n$ such that $x(T_1) = z$ and $I^f(T_1, T_2, x) = U^f(T_2 - T_1, z)$.*

Corollary 7.7. *For each $f \in \mathcal{A}$ and each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ there is an a.c. function $x : [T_1, T_2] \to R^n$ such that $I^f(T_1, T_2, x) = U^f(T_2 - T_1)$.*

Lemma 7.8 (Proposition 1.3.8 of [30]). *Let $f \in \mathcal{A}$, $0 < c_1 < c_2 < \infty$ and let $D, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{A}$ such that for each $g \in V$, each $T_1, T_2 \ge 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each a.c. function $x : [T_1, T_2] \to R^n$ satisfying $\min\{I^g(T_1, T_2, x),\ I^f(T_1, T_2, x)\} \le D$ the inequality $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \le \epsilon$ holds.*

Lemma 7.9 (Lemma 5.2.4 of [30]). *Let $f \in \mathcal{M}$ have (ATP) and let $\epsilon > 0$. Then there exists a number $q \ge 8$ such that for each $h_1, h_2 \in H(f)$ there exists an a.c. function $v : [0, q] \to R^n$ which satisfies*

$$v(0) = h_1,\ v(q) = h_2,\ \Gamma^f(0, q, v) \le \epsilon.$$

Proposition 7.10 (Proposition 8 of [33]). *Let $g \in \mathcal{M}$ possess (ATP) and $v : [0, \infty) \to R^n$ be an a.c. function such that $\sup\{|v(t)| : t \in [0, \infty)\} < \infty$,*

$$I^g(0, T, v) = U^g(0, T, v(0), v(T)) \text{ for all } T > 0.$$

Then the function v is (g)-perfect.

Proposition 7.11 (Theorem 1.2 of [31]). *Let $g \in \mathcal{L}$ and $v_1, v_2 : [0, \infty) \to R^n$ be (g)-perfect functions such that $v_1(0) = v_2(0)$. If there exist $t_1, t_2 \in [0, \infty)$ such that $(t_1, t_2) \ne (0, 0)$ and $v_1(t_1) = v_2(t_2)$, then $v_1(t) = v_2(t)$ for all $t \in [0, \infty)$.*

The following lemma is a particular case of Lemma 3.3 of [31].

Lemma 7.12. *Let $f \in \mathcal{A}$ have (ATP) and $h \in H(f)$. Then there exists an a.c. function $v : R^1 \to H(f)$ such that $v(0) = h$ and $\Gamma^f(-T, T, v) = 0$ for all $T > 0$.*

The following lemma is a particular case of Lemma 5.1 of [31].

Lemma 7.13. *Let $f \in \mathcal{L}$, $v_1, v_2 : [0, \infty) \to R^n$ be (f)-perfect functions, $0 \le T_1 < T_2$ and let $v_1(t) = v_2(t)$ for all $t \in [T_1, T_2]$. Then $v_1(t) = v_2(t)$ for all $t \in [0, \infty)$.*

Lemma 7.14 ([36])**.** *Let $f \in \mathcal{A}$ have (ATP) and $S_0 > 0$. Then there exist $K_0 > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$ and each $x \in R^n$ satisfying $|x| > K_0$ the inequality $\pi^g(x) > S_0$ holds.*

Lemma 7.15 ([36])**.** *Let $f \in \mathcal{M}$ have (ATP) and let $\epsilon > 0$. Then there exist numbers $q \ge 8$ and $\delta > 0$ such that for each $h_1, h_2 \in R^n$ satisfying $d(h_i, H(f)) \le \delta$, $i = 1, 2$ and each $T \ge q$ there exists an a.c. function $v : [0, T] \to R^n$ which satisfies $v(0) = h_1,\ v(T) = h_2,\ \Gamma^f(0, T, v) \le \epsilon$.*

8. Proof of Theorem 5.1

By Theorem 2.1, there exist a neighborhood $\mathcal{U}_1$ of f in $\mathcal{A}$ and $S_1 > 0$ such that for each $g \in \mathcal{U}_1$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:

(i) if an a. c. function $v : [T_1, T_2] \to R^n$ satisfies

$$|v(T_1)| \le M_1,\ I^g(T_1, T_2, v) \le U^g(T_2 - T_1, v(T_1)) + 1,$$

then $|v(t)| \le S_1$ for all $t \in [T_1, T_2]$;

(ii) if an a. c. function $v : [T_1, T_2] \to R^n$ satisfies

$$I^g(T_1, T_2, v) \le U^g(T_2 - T_1) + 1,$$

then $|v(t)| \le S_1$ for all $t \in [T_1, T_2]$.

In view of (5.2), there exist a neighborhood $\mathcal{V}_i$ of h_i, $i = 1, 2$ in $\mathfrak{A}$ and $S_2 > 0$ such that for all $\xi_i \in \mathcal{V}_i$, $i = 1, 2$ and each $z \in R^n$ satisfying $|z| \le S_1$,

$$|\xi_i(z)| \le S_2,\ i = 1, 2. \tag{8.1}$$

By Theorem 2.1, there exist a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of f in $\mathcal{A}$ and $S > S_1 + S_2$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:

(iii) if an a. c. function $v : [T_1, T_2] \to R^n$ satisfies

$$|v(T_1)| \le M_1,$$

$$I^g(T_1, T_2, v) \le U^g(T_2 - T_1, v(T_1)) + 1 + M_2 + 2S_2 + 2a_1,$$

then $|v(t)| \le S$ for all $t \in [T_1, T_2]$;

if an a. c. function $v : [T_1, T_2] \to R^n$ satisfies

$$I^g(T_1, T_2, v) \le U^g(T_2 - T_1) + 1 + M_2 + 2S_2 + 2a_1,$$

then $|v(t)| \le S$ for all $t \in [T_1, T_2]$.

Assume that

$$g \in \mathcal{U},\ \xi_i \in \mathcal{V}_i, i = 1, 2,\ T_1 \in [0, \infty),\ T_2 \ge T_1 + c \tag{8.2}$$

and an a. c. function $v : [T_1, T_2] \to R^n$ satisfies

$$|v(T_1)| \le M_1, \tag{8.3}$$

$$I^g(T_1, T_2, v) + \xi_1(v(T_2)) \le \sigma(g, \xi_1, v(T_1), T_1, T_2) + M_2 \tag{8.4}$$

or

$$I^g(T_1, T_2, v) + \xi_1(v(T_2)) + \xi_2(v(T_1)) \le \sigma(g, \xi_1, \xi_2, T_1, T_2) + M_2. \tag{8.5}$$

There exists an a. c. function $u : [T_1, T_2] \to R^n$ such that:
if (8.3) and (8.4) hold, then

$$u(T_1) = v(T_1),$$

$$I^g(T_1, T_2, u) \le U^g(T_2 - T_1, u(T_1)) + 1; \tag{8.6}$$

if (8.5) holds, then

$$I^g(T_1, T_2, u) \le U^g(T_2 - T_1) + 1. \tag{8.7}$$

It follows from (8.2), (8.4), (8.6), (8.7) and property (i) that

$$|u(t)| \le S_1, \; t \in [T_1, T_2]. \tag{8.8}$$

By (8.2), (8.8) and the choice of $\mathcal{V}_i$, $i = 1, 2$ (see (8.1)),

$$|\xi_1(u(T_2))|, \; |\xi_2(u(T_1))| \le S_2. \tag{8.9}$$

Assume that (8.3) and (8.4) hold. In view of (5.1), (8.2), (8.4), (8.6) and (8.9),

$$I^g(T_1, T_2, v) - a_1 \le I^g(T_1, T_2, v) + \xi_1(v(T_2))$$

$$\le M_2 + \sigma(g, \xi_1, v(T_1), T_1, T_2) \le M_2 + I^g(T_1, T_2, u) + \xi_1(u(T_2))$$

$$\le M_2 + S_2 + I^g(T_1, T_2, u) \le M_2 + S_2 + U^g(T_2 - T_1, v(T_1)) + 1,$$

$$I^g(T_1, T_2, v) \le U^g(T_2 - T_1, v(T_1)) + 1 + M_2 + S_2 + a_1. \tag{8.10}$$

Property (iii), (8.2), (8.3) and (8.10) imply that $|v(t)| \le S, \; t \in [T_1, T_2]$.
Assume that (8.5) holds. In view of (5.1), (8.2), (8.5), (8.7) and (8.9),

$$I^g(T_1, T_2, v) - 2a_1 \le I^g(T_1, T_2, v) + \xi_1(v(T_2)) + \xi_2(v(T_1))$$

$$\le M_2 + \sigma(g, \xi_1, \xi_2, T_1, T_2) \le M_2 + I^g(T_1, T_2, u) + \xi_1(u(T_2)) + \xi_2(u(T_1))$$

$$\le M_2 + 2S_2 + I^g(T_1, T_2, u) \le M_2 + 2S_2 + U^g(T_2 - T_1) + 1,$$

$$I^g(T_1, T_2, v) \le U^g(T_2 - T_1) + 1 + M_2 + 2S_2 + 2a_1. \tag{8.11}$$

Property (iii), (8.2) and (8.11) imply that $|v(t)| \le S, \; t \in [T_1, T_2]$. Theorem 5.1 is proved.

9. Proof of Theorem 5.3

By Theorems 1.2 and 2.1, there exist a neighborhood $\mathcal{U}_0 \subset \mathcal{U}_f$ of f in $\mathcal{A}$ and $L_0 \geq 1$ such that the following property holds:

(i) for each $g \in \mathcal{U}_0$, each $T \geq L_0$ and each a. c. function $u : [0,T] \to R^n$ which satisfies

$$|u(0)| \leq M_0,\ I^g(0,T,u) \leq U^g(T,u(0)) + 1$$

there exists $S_0 \in [0, L_0]$ such that $d(u(S_0), H(f)) \leq 1$.

By Theorem 2.2, there exist a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of f in $\mathcal{A}$ and numbers $l, S \geq 1$ and integers $L \geq 1$, $Q_* \geq L_0$ such that the following property holds:

(ii) for each $g \in \mathcal{U}$, each $T \geq L + lQ_*$ and each a.c. function $v : [0,T] \to R^n$ which satisfies at least one of the following conditions:

$$|v(0)| \leq M_0,\ I^g(0,T,v) \leq U^g(T,v(0)) + M_2 + M_1 + a_1;$$

$$I^g(0,T,v) \leq U^g(T) + 2M_2 + 2M_1 + 2a_1$$

the inequality $|v(t)| \leq S$ holds for all $t \in [0,T]$ and there exist sequences of numbers $\{b_i\}_{i=1}^{Q}$, $\{c_i\}_{i=1}^{Q} \subset [0,T]$ such that

$$Q \leq Q_*,\ 0 \leq c_i - b_i \leq l,\ i = 1, \dots, Q,$$

$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + L]\}) \leq \epsilon$ for each $\tau \in [0, T - L] \setminus \cup_{i=1}^{Q} [b_i, c_i]$.

Assume that

$$g \in \mathcal{U},\ T \geq L + lQ_*,\ h, \xi \in \mathfrak{A}, \tag{9.1}$$

$$|h(z)|,\ |\xi(z)| \leq M_2 \text{ for all } z \in R^n \text{ such that } |z| \leq S_f \tag{9.2}$$

and $v : [0,T] \to R^n$ is an a. c. function which satisfies at least one of the conditions (a), (b). By Corollaries 7.6 and 7.7, there exists an a. c. function $u : [0,T] \to R^n$ such that if condition (a) holds, then

$$u(0) = v(0), \tag{9.3}$$

$$I^g(0,T,u) = U^g(T, v(0)) \tag{9.4}$$

and if condition (b) holds, then

$$I^g(0,T,u) = U^g(T). \tag{9.5}$$

If condition (b) holds, then (9.1), (9.5) and property (P1) imply that

$$|u(t)| \leq S_f,\ t \in [0,T]. \tag{9.6}$$

Assume that condition (a) holds. We show that

$$|u(T)| \leq S_f.$$

By condition (a), property (i) and (9.1)-(9.5), there exists $S_0 \in [0, L_0]$ such that

$$d(u(S_0), H(f)) \leq 1. \tag{9.7}$$

Property (P2) applied to the function $u(t + S_0)$, $t \in [0, T - S_0]$, (9.1), (9.4) and (9.7) imply that

$$|u(t)| \leq S_f,\ t \in [S_0, T]. \tag{9.8}$$

Together with (9.6) this implies that in both cases

$$|u(T)| \le S_f. \tag{9.9}$$

If condition (a) holds, then by (5.1), (9.1)-(9.4) and (9.9),

$$I^g(0,T,v) - a_1 \le I^g(0,T,v) + h(v(T))$$
$$\le M_1 + \sigma(g,h,v(0),0,T) \le M_1 + I^g(0,T,u) + h(u(T))$$
$$\le M_2 + M_1 + I^g(0,T,u) \le M_2 + M_1 + U^g(T,v(0)),$$

$$I^g(0,T,v) \le U^g(T,v(0)) + M_2 + M_1 + a_1. \tag{9.10}$$

If condition (b) holds, then by (5.1), (9.1), (9.2), (9.5) and (9.6),

$$I^g(0,T,v) - 2a_1 \le I^g(0,T,v) + h(v(T)) + \xi(v(0))$$
$$\le M_1 + \sigma(g,h,\xi,0,T) \le M_1 + I^g(0,T,u) + h(u(T)) + \xi(u(0))$$
$$\le 2M_2 + M_1 + I^g(0,T,u) \le 2M_2 + M_1 + U^g(T),$$

$$I^g(0,T,v) \le U^g(T) + 2M_2 + M_1 + 2a_1. \tag{9.11}$$

Property (ii), (9.1), (9.10) and (9.11) imply that he inequality $|v(t)| \le S$ holds for all $t \in [0,T]$ and there exist sequences of numbers

$$\{b_i\}_{i=1}^q,\ \{c_i\}_{i=1}^q \subset [0,T]$$

such that $q \le Q_*,\ 0 \le c_i - b_i \le l,\ i = 1, \dots, q$ and that

$$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + L]\}) \le \epsilon$$

for each $\tau \in [0, T-L] \setminus \cup_{i=1}^q [b_i, c_i]$. This completes the proof of Theorem 5.3.

10. Proof of Theorem 5.4

By Theorem 1.3, there exist a neighborhood $\mathcal{U}_1$ of f in $\mathcal{A}$ and numbers $l_0 > l > 0$, $\delta > 0$ such that the following property holds:

(i) for each $g \in \mathcal{U}_1$, each $T \ge 2l_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$d(v(0), H(f)) \le \delta,\ d(v(T), H(f)) \le \delta,$$
$$I^g(0,T,v) \le U^g(0,T,v(0),v(T)) + \delta$$

the inequality $\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \le \epsilon$ holds for each $\tau \in [0, T-l]$.

By Theorem 1.2, there exist a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of f in $\mathcal{A}$ and numbers $M_1 > M_0$, $\tilde{l}_0 > 2l_0 + 1$ such that the following property holds:

(ii) for each $g \in \mathcal{U}$, each $T \ge \tilde{l}_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \le M_0,\ I^g(0,T,v) \le U^g(0,T,v(0),v(T)) + M_0$$

the inequality $|v(t)| \le M_1$ holds for all $t \in [0,T]$ and there exists a set $E \subset [0,T]$ which is a finite union of closed subintervals of $[0,T]$ such that $\mathrm{mes}(E) < \tilde{l}_0$ and for each $t \in [0,T] \setminus E$, $d(v(t), H(f)) \le \delta$.

Set

$$l_1 = 8\tilde{l}_0 + 2. \tag{10.7}$$

Assume that

$$g \in \mathcal{U},\ T \geq 2l_1 + l \tag{10.8}$$

and that an a.c. function $v : [0, T] \to R^n$ satisfies

$$|v(0)|, |v(T)| \leq M_0, \tag{10.9}$$

$$I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + M_0 \tag{10.10}$$

and that for each $S \in [0, T - l_1]$,

$$I^g(S, S + l_1, v) \leq U^g(S, S + l_1, v(S), v(S + l_1)) + \delta. \tag{10.11}$$

Property (ii) and (10.7)-(10.10) imply that

$$|v(t)| \leq M_1,\ t \in [0, T] \tag{10.12}$$

and that that the following property holds:

(iii) for each $\tau \in [0, T - \tilde{l}_0]$ there exists $t \in [\tau, \tau + \tilde{l}_0]$ such that

$$d(v(t), H(f)) \leq \delta.$$

Set

$$\tau_1 = \inf\{t \in [0, T] :\ d(v(t), H(f)) \leq \delta\},$$

$$\tau_2 = \sup\{t \in [0, T] :\ d(v(t), H(f)) \leq \delta\}. \tag{10.13}$$

Property (ii), (10.7), (10.8) and (10.13) imply that

$$d(v(\tau_i), H(f)) \leq \delta,\ i = 1, 2, \tag{10.14}$$

$$\tau_1 \leq \tilde{l}_0, \tau_2 \geq T - \tilde{l}_0, \tag{10.15}$$

$$\tau_2 - \tau_1 \geq T - 2\tilde{l}_0 \geq 2l_1 - 2\tilde{l}_0 \geq 12\tilde{l}_0 + 4. \tag{10.16}$$

Assume that

$$S \in [\tau_1, \tau_2 - l]. \tag{10.17}$$

There are the following cases:

$$S - 2\tilde{l}_0 < \tau_1,\ S + l + 2\tilde{l}_0 > \tau_2; \tag{10.18}$$

$$S - 2\tilde{l}_0 < \tau_1,\ S + l + 2\tilde{l}_0 \leq \tau_2; \tag{10.19}$$

$$S - 2\tilde{l}_0 \geq \tau_1,\ S + l + 2\tilde{l}_0 > \tau_2; \tag{10.20}$$

$$S - 2\tilde{l}_0 \geq \tau_1,\ S + l + 2\tilde{l}_0 \leq \tau_2; \tag{10.21}$$

If (10.18) holds, then in view of (10.16),

$$12\tilde{l}_0 + 4 \leq \tau_2 - \tau_1 < S + l + 2\tilde{l}_0 - (S - 2\tilde{l}_0) = 4\tilde{l}_0 + l,$$

a contradiction. Therefore (10.18) does not hold.

If (10.19) holds, then we set $S_1 = \tau_1$, property (iii) implies that there exists $S_2 \in [S + l + \tilde{l}_0, S + l + 2\tilde{l}_0]$ such that $d(v(S_2), H(f)) \leq \delta$ and in view of (10.14), (10.17) and (10.19),

$$d(v(S_i), H(f)) \leq \delta,\ i = 1, 2, \tag{10.22}$$

(10.23) $$2l_0 + 1 + l < l + \tilde{l}_0 \le S_2 - S_1 \le l + 4\tilde{l}_0.$$

If (10.20) holds, then we set $S_2 = \tau_2$, property (iii) implies that there exists $S_1 \in [S - 2\tilde{l}_0, S - \tilde{l}_0]$ such that $d(v(S_1), H(f)) \le \delta$ and in view of (10.14), (10.17) and (10.20), relation (10.22) is true and

$$\tilde{l}_0 + l \le S_2 - S_1 \le l + 4\tilde{l}_0.$$

If (10.21) holds, then property (iii) implies that there exist

$$S_1 \in [S - 2\tilde{l}_0, S - \tilde{l}_0],\ S_2 \in [S + l + \tilde{l}_0, S + l + 2\tilde{l}_0]$$

such that (10.22) holds. Thus in all the cases $S_1, S_2 \in [\tau_1, \tau_2]$ and (10.22) and (10.23) are true.

By (10.7), (10.11) and (10.23),

(10.24) $$I^g(S_1, S_2, v) \le U^g(S_1, S_2, v(S_1), v(S_2)) + \delta.$$

It follows from (10.8), (10.17), (10.22)-(10.24), the choice of S_1, S_2 and property (i) that $\mathrm{dist}(H(f), \{v(t) :\ t \in [S, S + l]\}) \le \epsilon$. Theorem 5.4 is proved.

11. Proof of Theorem 5.6

By Theorem 5.4, there exist a neighborhood $\mathcal{U}_1$ of f in $\mathcal{A}$ and numbers $l_1 > l > 0$, $\delta \in (0, 1]$ such that the following property holds:

(i) for each $g \in \mathcal{U}_1$, each $T \ge 2l_1 + l$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$d(v(0), H(f)) \le \delta,\ d(v(T), H(f)) \le \delta,$$
$$I^g(0, T, v) \le U^g(0, T, v(0), v(T)) + M_1$$

and such that for each $S \in [0, T - l_1]$,

$$I^g(S, S + l_1, v) \le U^g(S, S + l_1, v(S), v(S + l_1)) + \delta$$

the inequality

(11.1) $$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \le \epsilon$$

holds for all $\tau \in [0, T - l]$.

By Theorem 5.3, there exist a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of f in $\mathcal{A}$ and numbers $S > 0$, $L_0 > 2l_1 + 1$ such that the following property holds:

(ii) for each $g \in \mathcal{U}$, each $h, \xi \in \mathfrak{A}$ satisfying

(11.2) $$|h(z)|,\ |\xi(z)| \le M_2 \text{ for all } z \in R^n \text{ such that } |z| \le S_f,$$

each $T \ge L_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies at least one of the following conditions:

$$|v(0)| \le M_0,$$
$$I^g(0, T, v) + h(v(T)) \le \sigma(g, h, v(0), 0, T) + M_1;$$
$$I^g(0, T, v) + h(v(T)) + \xi(v(0)) \le \sigma(g, h, \xi, 0, T) + M_1$$

the inequality $|v(t)| \le S$ holds for all $t \in [0, T]$ and there exist a set $E \subset [0, T]$ which is a finite union of closed subintervals of $[0, T]$ such that $\mathrm{mes}(E) < L_0$ and for each $t \in [0, T] \setminus E$, $d(v(t), H(f)) \le \delta$.

Set

$$L = 4L_0. \tag{11.3}$$

Assume that $g \in \mathcal{U}$, $h, \xi \in \mathfrak{A}$, (11.2) and (11.4) hold, $T \geq 2L + l$ and that an a. c. function $v : [0, T] \to R^n$ satisfies at least one of conditions (a), (b) and that for each $\tau \in [0, T - L]$,

$$I^g(\tau, \tau + L, v) \leq U^g(\tau, \tau + L, v(\tau), v(\tau + L)) + \delta. \tag{11.5}$$

By property (ii), conditions (a),(b) and (11.2)-(11.4) , $|v(t)| \leq S,\ t \in [0, T]$ and there exist

$$\tau_1 \in [0, L_0],\ \tau_2 \in [T - L_0, T] \tag{11.6}$$

such that

$$d(v(\tau_i), H(f)) \leq \delta,\ i = 1, 2. \tag{11.17}$$

If $d(v(0), H(f)) \leq \delta$, then $\tau_1 = 0$ and if $d(v(T), H(f)) \leq \delta$, then $\tau_2 = T$. Property (i), conditions (a), (b), (11.3), (11.4), (11.6) and (11.7) imply that for each $\tau \in [\tau_1, \tau_2 - l]$, (11.1) is true. Theorem 5.6 is proved.

12. An auxiliary result for Theorem 6.2 and 6.3

Suppose that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property and that $h \in \mathfrak{A}$. Set

$$\mathcal{D}(f, h) = \{z \in R^n : (\pi^f + h)(z) = \inf(\pi^f + h)\}. \tag{12.1}$$

Lemma 12.1. *Let $\epsilon \in (0, 1)$ and $T_0 > 0$. Then there exists $\delta \in (0, \epsilon)$ such that for each a.c. function $u : [0, T_0] \to R^n$ which satisfies*

$$(\pi^f + h)(u(0)) \leq \inf(\pi^f + h) + \delta,\ \Gamma^f(0, T_0, u) \leq \delta$$

there exists an (f)-perfect function $w : [0, \infty) \to R^n$ such that

$$(\pi^f + h)(w(0)) = \inf(\pi^f + h),\ |u(t) - w(t)| \leq \epsilon \text{ for all } t \in [0, T_0].$$

Proof. Assume that the lemma does not hold. Then there exist a sequence $\{\delta_k\}_{k=1}^\infty \subset (0, 1)$ and a sequence of a. c. functions $u_k : [0, T_0] \to R^n$, $k = 1, 2, \dots$ such that

$$\lim_{k \to \infty} \delta_k = 0 \tag{12.2}$$

and that for each integer $k \geq 1$ and each

$$w \in \cup\{\mathcal{P}(f, z) : z \in \mathcal{D}(f, h)\}, \tag{12.3}$$

we have

$$(\pi^f + h)(u_k(0)) \leq \inf(\pi^f + h) + \delta_k,\ \Gamma^f(0, T_0, u_k) \leq \delta_k, \tag{12.4}$$

$$\sup\{|u_k(t) - w(t)| : t \in [0, T_0]\} > \epsilon. \tag{12.5}$$

By (12.2), (12.4) and Proposition 6.1, extracting a subsequence and re-indexing if necessary we may assume without loss of generality that the sequence $\{u_k(0)\}_{k=1}^\infty$ converges and

$$\lim_{k \to \infty} u_k(0) \in \mathcal{D}(f, h). \tag{12.6}$$

Let $k \geq 1$ be an integer. By Proposition 3.6, there exists an (f)-good and (f)-perfect function $y_k : [0, \infty) \to R^n$ such that

$$y_k(0) = u_k(T_0). \tag{12.7}$$

In view of (12.7) there exists an a. c. function $v_k : [0, \infty) \to R^n$ such that

$$v_k(t) = u_k(t),\ t \in [0, T_0],\ v_k(t) = y_k(t - T_0),\ t \in (T_0, \infty). \tag{12.8}$$

Since the function y_k is (f)-perfect it follows from (12.4) and (12.8) that for any $T > 0$,

$$\Gamma^f(0, T, v_k) \leq \Gamma^f(0, T_0, u_k) \leq \delta_k. \tag{12.9}$$

In view of Propositions 1.1 and 3.5, the function v_k is (f)-good. By Proposition 7.3, there exists a number $S_1 > 0$ such that

$$\limsup_{t \to \infty} |v_k(t)| \leq S_1 \text{ for all integers } k \geq 1. \tag{12.10}$$

It follows from (8.23), (12.6), (12.9) and Theorem 2.1 that

$$\sup\{\sup\{|u_k(t)| :\ t \in [0, T_0]\} :\ k = 1, 2, \dots\} < \infty. \tag{12.11}$$

By (3.6) and (12.4), for each natural number k,

$$I^f(0, T_0, u_k) = \Gamma^f(0, T_0, u_k) + T_0\mu(f) + \pi^f(u_k(0)) - \pi^f(u_k(T_0))$$

$$\leq \delta_k + T_0\mu(f) + \pi^f(u_k(0)) - \pi^f(u_k(T_0)). \tag{12.12}$$

By (12.11), (12.12) and the continuity of π^f, the sequence $\{I^f(0, T_0, u_k)\}_{k=1}^{\infty}$ is bounded. By Proposition 7.4, extracting subsequences we can show the existence of a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ and an a.c. function $u : [0, T_0] \to R^n$ such that

$$I^f(0, T_0, u) \leq \liminf_{k \to \infty} I^f(0, T_0, u_k), \tag{12.13}$$

$$u_{i_k}(t) \to u(t) \text{ as } k \to \infty \text{ uniformly on } [0, T_0]. \tag{12.14}$$

In view of (12.6) and (12.14),

$$u(0) \in \mathcal{D}(f, h). \tag{12.15}$$

By (3.6), (12.2), (12.4), (12.13), (12.14) and the continuity of π^f,

$$\Gamma^f(0, T_0, u) = I^f(0, T_0, u) - \pi^f(u(0)) + \pi^f(u(T_0)) - T_0\mu(f)$$

$$\leq \liminf_{k \to \infty} I^f(0, T_0, u_{i_k}) - \lim_{k \to \infty} \pi^f(u_{i_k}(0)) + \lim_{k \to \infty} \pi^f(u_{i_k}(T_0)) - T_0\mu(f)$$

$$= \liminf_{k \to \infty} [I^f(0, T_0, u_{i_k}) - \pi^f(u_{i_k}(0)) + \pi^f(u_{i_k}(T_0)) - T_0\mu(f)]$$

$$\leq \liminf_{k \to \infty} \Gamma^f(0, T_0, u_{i_k}) \leq \lim_{k \to \infty} \delta_{i_k} = 0.$$

Together with (3.7) this implies that $\Gamma^f(0, T_0, u) = 0$. Combined with Proposition 3.6 this implies that there exists an (f)-perfect function $\tilde{u} : [0, \infty) \to R^n$ such that

$$\tilde{u}(t) = u(t),\ t \in [0, T_0]. \tag{12.16}$$

It follows from (12.15) and (12.16) that

$$\tilde{u} \in \cup\{\mathcal{P}(f, z) :\ z \in \mathcal{D}(f, h)\}. \tag{12.17}$$

By (12.14) and (12.16), for all sufficiently large natural numbers k,

$$\sup\{|u_{i_k}(t) - \tilde{u}(t)| : \ t \in [0, T_0]\} \leq \epsilon/2.$$

Together with (12.17) this contradicts (12.5) which holds for any integer $k \geq 1$ and any function w satisfying (12.3). The contradiction we have reached proves Lemma 12.1. □

13. Proof of Theorem 6.2

By Lemma 12.1, there exist $\delta_0 \in (0, \epsilon)$ such that the following property holds:

(i) for each a.c. function $u : [0, L_0] \to R^n$ which satisfies

$$(\pi^{\bar{f}} + h)(u(0)) \leq \inf((\pi^{\bar{f}} + h) + 8\delta_0, \ \Gamma^{\bar{f}}(0, \tau_0, u) \leq 4\delta_0$$

there exists a $(\bar{f})$-perfect function $w : [0, \infty) \to R^n$ such that

$$(\pi^{\bar{f}} + h)(w(0)) = \inf((\pi^{\bar{f}} + h), \ |u(t) - w(t)| \leq \epsilon \text{ for all } t \in [0, L_0].$$

By Lemma 7.15, there exist a number $q \geq 8$ and $\delta_1 \in (0, \delta_0)$ such that the following property holds:

(ii) for each $z_1, z_2 \in R^n$ satisfying $d(z_i, H(f)) \leq \delta_1$, $i = 1, 2$ and each $T \geq q$ there exists an a.c. function $u : [0, T] \to R^n$ which satisfies

$$u(0) = z_1, \ u(T) = z_2, \ \Gamma^{\bar{f}}(0, T, u) \leq \delta_0.$$

By Propositions 3.6 and 6.1, there exists an $(\bar{f})$-perfect function $w_* : [0, \infty) \to R^n$ such that

$$(\pi^{\bar{f}} + h)(w_*(0)) = \inf(\pi^{\bar{f}} + h). \tag{13.1}$$

Proposition 4.1 implies that

$$\sup\{|w_*(t)| : \ t \in [0, \infty)\} < \infty. \tag{13.2}$$

It follows from (ATP) and (13.2) that there exists $S_0 > 0$ such that

$$d(w_*(t), H(f)) \leq \delta_1 \text{ for all } t \geq S_0. \tag{13.3}$$

By Theorem 5.6, there exist a neighborhood $\mathcal{U}_1$ of f in $\mathcal{A}$, a neighborhood $\mathcal{V}_1$ of h in $\mathfrak{A}$ and numbers $L_1 > l_1 > 0$, $S_1 > 0$, $\delta \in (0, \delta_1)$ such that the following property holds:

(iii) for each $g \in \mathcal{U}_1$, each $\xi \in \mathcal{V}_1$, each $T \geq 2L_1 + l_1$ and each a.c. function $v : [0, T] \to R^n$ which satisfies $|v(0)| \leq M$,

$$I^g(0, T, v) + \xi(v(T)) \leq \sigma(g, \xi, v(0), 0, T) + \delta$$

we have $|v(t)| \leq S_1, \ t \in [0, T]$, $\mathrm{dist}(H(f), \{v(t) : \ t \in [\tau, \tau + l_1]\}) \leq \delta_1$ for each $\tau \in [L_1, T - L_1 - l_1]$.

Fix

$$T_0 \geq 4(L_1 + L_0 + q + S_0). \tag{13.4}$$

By Lemma 7.8 there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of f in $\mathcal{A}$ such that the following property holds:

(iv) for each $g \in \mathcal{U}$, each $\tau \geq 0$ and each a.c. function $u : [\tau, \tau + S_0 + L_0 + L_1 + q] \to R^n$ satisfying

$$\min\{I^f(\tau, \tau + S_0 + L_0 + L_1 + q, u), \ I^g(\tau, \tau + S_0 + L_0 + L_1 + q, u)\}$$

$$\leq a_1 + q + 8 + |\pi^{\bar{f}}(w_*(0))| + \sup\{|\pi^{\bar{f}}(z)| : \ z \in R^n \text{ and } |z| \leq S_1\}$$
$$+|\mu(f)|(S_0 + L_0 + L_1 + q) + |h(w_*(0))|$$

we have

$$|I^f(\tau, \tau + S_0 + L_0 + L_1 + q, u) - I^g(\tau, \tau + S_0 + L_0 + L_1 + q, u)| \leq \delta_0.$$

Clearly, there exists a neighborhood $\mathcal{V} \subset \mathcal{V}_1$ of h in $\mathfrak{A}$ such that the following property holds:

(v) for each $\xi \in \mathcal{V}$,

$$|\xi(z) - h(z)| \leq \delta_0$$

for all $z \in R^n$ such that $|z| \leq S_1 + |w_*(0)|$.

Assume that

$$g \in \mathcal{U}, \ \xi \in \mathcal{V}, \ T \geq T_0 \tag{13.5}$$

and that an a. c. function $v : [0, T] \to R^n$ satisfies

$$|v(0)| \leq M_1 \tag{13.6}$$

$$I^g(0, T, v) + \xi(v(T)) \leq \sigma(g, \xi, v(0), 0, T) + \delta. \tag{13.7}$$

Property (iii) and (13.4)-(13.7) imply that

$$d(v(t), H(f)) \leq \delta_1, \ t \in [L_1, T - L_1], \tag{13.8}$$

$$|v(t)| \leq S_1, \ t \in [0, T]. \tag{13.9}$$

In view of (13.3),

$$d(w_*(S_0 + L_0), H(f)) \leq \delta_1. \tag{13.10}$$

By (13.4), (13.5) and (13.8),

$$d(v(T - S_0 - L_0 - q - L_1), H(f)) \leq \delta_1. \tag{13.11}$$

Property (ii), (13.4), (13.5), (13.10) and (13.11) imply that there exists an a. c. function $w_1 : [S_0 + L_0, S_0 + L_0 + q + L_1] \to R^n$ such that

$$\Gamma^{\bar{f}}(S_0 + L_0, S_0 + L_0 + q + L_1, w_1) \leq \delta_0, \tag{13.12}$$

$$w_1(S_0 + L_0) = w_*(S_0 + L_0), \ w_1(S_0 + L_0 + q + L_1) = v(T - S_0 - L_0 - q - L_1). \tag{13.13}$$

Set

$$w_2(t) = w_*(t), \ t \in [0, S_0 + L_0], \tag{13.14}$$

$$w_2(t) = w_1(t), \ t \in (S_0 + L_0, S_0 + L_0 + q + L_1]. \tag{13.15}$$

Clearly, $w_2 : [0, S_0 + L_0 + q + L_1] \to R^n$ is an a. c. function. By (13.12), (13.14), (13.15) and the choice of w_*,

$$\Gamma^{\bar{f}}(0, S_0 + L_0 + q + L_1, w_2)$$

$$= \Gamma^{\bar{f}}(0, S_0 + L_0, w_*) + \Gamma^{\bar{f}}(S_0 + L_0, S_0 + L_0 + q + L_1, w_1) \leq \delta_0. \tag{13.16}$$

Set

$$\widehat{w}_2(t) = w_2(T - t), \ t \in [T - (S_0 + L_0 + q + L_1), T]. \tag{13.17}$$

It follows from (13.13), (13.15) and (13.17) that

$$w_2(T-(S_0+L_0+q+L_1)) = v(T-S_0-L_0-L_1-q). \tag{13.18}$$

By (13.7) and (13.18),

$$\begin{aligned}&I^g(T-S_0-L_0-L_1-q,T,v)+\xi(v(T))\\ &\leq I^g(T-S_0-L_0-L_1-q,T,\widehat{w}_2)+\xi(\widehat{w}_2(T))+\delta.\end{aligned} \tag{13.19}$$

It follows from (3.3), (3.6) and (3.13)-(3.17) that

$$\begin{aligned}I^f(T-S_0-L_0-L_1-q,T,\widehat{w}_2) &= I^{\bar f}(0,S_0+L_0+L_1+q,w_2)\\ &= \Gamma^{\bar f}(0,S_0+L_0+L_1+q,w_2)+\pi^{\bar f}(w_2(0))\\ &\quad -\pi^{\bar f}(w_2(S_0+L_0+L_1+q))+(S_0+L_0+L_1+q)\mu(f)\\ &\leq \delta_0+\pi^{\bar f}(w_2(0))-\pi^{\bar f}(w_2(S_0+L_0+L_1+q))+(S_0+L_0+L_1+q)\mu(f)\\ &= \delta_0+\pi^{\bar f}(w_*(0))-\pi^{\bar f}(v(T-S_0-L_0-L_1-q))+(S_0+L_0+L_1+q)\mu(f).\end{aligned} \tag{13.20}$$

In view of (13.9) and (13.20),

$$\begin{aligned}&I^f(T-S_0-L_0-L_1-q,T,\widehat{w}_2)\\ &\leq \delta_0+|\pi^{\bar f}(w_*(0))|+\sup\{|\pi^{\bar f}(z)|:\ z\in R^n,\ |z|\leq S_1\}+(S_0+L_0+L_1+q)|\mu(f)|.\end{aligned} \tag{13.21}$$

Property (iv), (13.5) and (13.21) imply that

$$|I^f(T-S_0-L_0-L_1-q,T,\widehat{w}_2)-I^g(T-S_0-L_0-L_1-q,T,\widehat{w}_2)|\leq \delta_0. \tag{13.22}$$

By (13.14) and (13.17),

$$\widehat{w}_2(T)=w_2(0)=w_*(0). \tag{13.23}$$

Property (v), (13.5) and (13.23) imply that

$$|h(\widehat{w}_2(T))-\xi(\widehat{w}_2(T))|\leq \delta_0. \tag{13.24}$$

It follows from (13.19), (13.20) and (13.22)-(13.24) that

$$\begin{aligned}&I^g(T-S_0-L_0-L_1-q,T,v)+\xi(v(T))\\ &\leq I^f(T-S_0-L_0-L_1-q,T,\widehat{w}_2)+h(\widehat{w}_2(T))+3\delta_0\\ &\leq \delta_0+\pi^{\bar f}(w_*(0))-\pi^{\bar f}(v(T-S_0-L_0-L_1-q))\\ &\quad +(S_0+L_0+L_1+q)\mu(f)+h(w_*(0))+3\delta_0.\end{aligned} \tag{13.25}$$

Property (v) and (13.9) imply that

$$|\xi(v(T))-h(v(T))|\leq \delta_0. \tag{13.26}$$

By (5.1), (13.9) and (13.25),

$$\begin{aligned}I^g(T-S_0-L_0-L_1-q,T,v)&\leq a_1+4\delta_0+|\pi^{\bar f}(w_*(0))|+|h(w_*(0))|\\ &\quad +\sup\{|\pi^{\bar f}(z)|:\ z\in R^n,\ |z|\leq S_1\}+(S_0+L_0+L_1+q)|\mu(f)|.\end{aligned} \tag{13.27}$$

Property (iv), (13.5) and (13.27) imply that

$$|I^g(T-S_0-L_0-L_1-q,T,v)-I^f(T-S_0-L_0-L_1-q,T,v)|\leq \delta_0. \tag{13.28}$$

It follows from (13.25), (13.26) and (13.28) that

$$I^f(T - S_0 - L_0 - L_1 - q, T, v) + h(v(T))$$
$$\le I^g(T - S_0 - L_0 - L_1 - q, T, v) + \xi(v(T)) + 2\delta_0$$
$$\le 6\delta_0 + \pi^{\bar{f}}(w_*(0)) - \pi^{\bar{f}}(v(T - S_0 - L_0 - L_1 - q))$$
$$+(S_0 + L_0 + L_1 + q)\mu(f) + h(w_*(0)). \tag{13.29}$$

Set

$$\widehat{v}(t) = v(T - t),\ t \in [0, T]. \tag{13.30}$$

By (3.3), (13.29) and (13.30),

$$I^{\bar{f}}(0, S_0 + L_0 + L_1 + q, \widehat{v}) + h(\widehat{v}(0))$$
$$= I^f(T - S_0 - L_0 - L_1 - q, T, v) + h(v(T))$$
$$\le 6\delta_0 + \pi^{\bar{f}}(w_*(0)) - \pi^{\bar{f}}(v(T - S_0 - L_0 - L_1 - q))$$
$$+(S_0 + L_0 + L_1 + q)\mu(f) + h(w_*(0)). \tag{13.31}$$

In view of (3.6) and (13.31),

$$\Gamma^{\bar{f}}(0, S_0 + L_0 + L_1 + q, \widehat{v}) + \pi^{\bar{f}}(\widehat{v}(0)) + h(\widehat{v}(0))$$
$$-\pi^{\bar{f}}(v(T - S_0 - L_0 - L_1 - q)) + (S_0 + L_0 + L_1 + q)\mu(f)$$
$$\le 6\delta_0 + \pi^{\bar{f}}(w_*(0)) - \pi^{\bar{f}}(v(T - S_0 - L_0 - L_1 - q))$$
$$+(S_0 + L_0 + L_1 + q)\mu(f) + h(w_*(0))$$

and

$$\Gamma^{\bar{f}}(0, S_0 + L_0 + L_1 + q, \widehat{v}) + (\pi^{\bar{f}} + h)(\widehat{v}(0)) \le 6\delta_0 + (\pi^{\bar{f}} + h)(w_*(0)). \tag{13.32}$$

By (3.6), (3.7), (13.1) and (13.32),

$$\Gamma^{\bar{f}}(0, L_0, \widehat{v}) \le \Gamma^{\bar{f}}(0, S_0 + L_0 + L_1 + q, \widehat{v}) \le 6\delta_2,$$
$$(\pi^{\bar{f}} + h)(\widehat{v}(0)) \le 6\delta_0 + (\pi^{\bar{f}} + h)(w_*(0)) = \inf(\pi^{\bar{f}} + h) + 6\delta_0.$$

The inequalities above and property (i) imply that there exists a $(\bar{f})$-perfect function $w : [0, \infty) \to R^n$ such that

$$(\pi^{\bar{f}} + h)(w(0)) = \inf((\pi^{\bar{f}} + h) \tag{13.33}$$

and

$$|\widehat{v}(t) - w(t)| \le \epsilon \text{ for all } t \in [0, L_0]. \tag{13.34}$$

By (13.30) and (13.34), for all $t \in [0, L_0]$, $|w(t) - v(T - t)| \le \epsilon$. Theorem 6.2 is proved.

14. Proof of Theorem 6.3

Theorem 6.2 and Proposition 5.2 imply the following result.

Theorem 14.1. *Suppose that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property and that $h \in \mathfrak{A}$. Let $\epsilon, M, L_0 > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, a neighborhood $\mathcal{V}$ of h in $\mathfrak{A}$ and numbers $\delta \in (0, \epsilon)$ and $T_0 > L_0$ such that for each $T \geq T_0$, each $g \in \mathcal{U}$, each $\xi \in \mathcal{V}$ and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(T)| \leq M,$$

$$I^g(0, T, v) + \xi(v(0)) \leq \widehat{\sigma}(g, \xi, v(T), 0, T) + \delta$$

there exists an (f)-perfect function $w : [0, \infty) \to R^n$ such that

$$(\pi^f + h)(w(0)) = \inf(\pi^f + h),$$

$$|v(t) - w(t)| \leq \epsilon \text{ for all } t \in [0, L_0].$$

Theorem 6.3 easily follows from Theorems 5.1, 6.2 and 14.1.

15. Auxiliary results for Theorems 6.5 and 6.8

Let $\mathfrak{M}$ be one of the following spaces: $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\bar{\mathcal{M}}_q$, $q \geq 3$ is an integer. If $\mathfrak{M} = \bar{\mathcal{M}}_q$, where $q \geq 3$ is an integer, then we set $\tilde{\mathcal{M}} = \mathcal{M}_q$; if $\mathfrak{M} = \mathcal{L}_q$, where $q \in \{0, 1, 2\}$, then we set $\tilde{\mathcal{M}} = \mathcal{L}$. Denote by E_0 the set of all $f \in \tilde{\mathcal{M}}$ which has (ATP).

Lemma 15.1 ([36])**.** *The set E_0 is an everywhere dense subset of $\mathfrak{M}$.*

This lemma was proved on page 170 of [30]. The following result was proved in Section 3 of Chapter 2 of [3] (see also Proposition 3.7.1 of [30]).

Lemma 15.2. *Let Ω be a closed subset of R^s. Then there exists a bounded nonnegative function $\phi \in C^\infty(R^s)$ such that $\Omega = \{x \in R^s : \phi(x) = 0\}$ and for each sequence of nonnegative integers $p_1, \dots, p_s$, the function $\partial^{|p|}\phi/\partial x_1^{p_1} \dots \partial x_s^{p_s} : R^s \to R^1$ is bounded, where $|p| = \sum_{i=1}^s p_i$.*

For each $h \in \mathfrak{A}$ denote by E_h the set of all $f \in E_0$ such that there exists a a unique (f)-perfect function $v : [0, \infty) \to R^n$ such that

$$(\pi^f + h)(v(0)) = \inf(\pi^f + h).$$

Lemma 15.3. *Let $f \in E_0$, $h \in \mathfrak{A}$,*

$$\inf\{(\pi^f + h)(z) : z \in H(f)\} > \inf(\pi^f + h) \tag{15.1}$$

and let $\mathcal{V}$ be a neighborhood of f in $\mathfrak{M}$. Then $\mathcal{V} \cap E_h \neq \emptyset$.

Proof. Let $z_0 \in R^n$ satisfy

$$(\pi^f + h)(z_0) = \inf(\pi^f + h). \tag{15.2}$$

By Proposition 3.6, there exists an (f)-good and (f)-perfect function $v : [0, \infty) \to R^n$ for which

$$v(0) = z_0. \tag{15.3}$$

In view of (ATP),

$$\Omega(v) = H(f). \tag{15.4}$$

Together with (15.1) and (15.3) this implies that there exists $\epsilon > 0$ such that for all sufficiently large positive numbers t,

$$(\pi^f + h)(z_0) + \epsilon < (\pi^f + h)(v(t)).$$

Therefore there is a number $\tau_0 \geq 0$ such that

$$(\pi^f + h)(v(\tau_0)) = (\pi^f + h)(z_0), \tag{15.5}$$

$$(\pi^f + h)(v(t)) > (\pi^f + h)(z_0) \text{ for all } t > \tau_0. \tag{15.6}$$

We may assume without loss of generality that $\tau_0 = 0$. Then

$$(\pi^f + h)(v(t)) > (\pi^f + h)(z_0) \text{ for all } t > 0. \tag{15.7}$$

Since the function v is (f)-perfect it follows from (3.6), (15.3), (15.4) and Lemma 7.1 that

$$\liminf_{T\to\infty}[I^f(0, T, v) - T\mu(f)] = \liminf_{T\to\infty}[\pi^f(v(0)) - \pi^f(v(T))] = \pi^f(z_0). \tag{15.8}$$

By Lemma 15.2 and (15.4), there exists a bounded nonnegative function $\phi \in C^\infty(R^n)$ such that the function $\partial^{|p|}\phi/\partial x_1^{p_1} \dots \partial x_n^{p_n} : R^n \to R^1$ is bounded, for each sequence of nonnegative integers $p_1, \dots, p_n$, where $|p| = \sum_{i=1}^n p_i$ and

$$\{x \in R^n : \phi(x) = 0\} = H(f) \cup \{v(t) : t \in [0, \infty)\}. \tag{15.9}$$

For any $r \in (0, 1)$ define a function $f_r : R^n \times R^n \to R^1$ by

$$f_r(x, y) = f(x, y) + r\phi(x), \ x, y \in R^n.$$

Arguing as in the proof of Lemma 10.3 of [36] we can show that for any $r \in (0, 1)$, $f_r \in \tilde{\mathcal{M}}$,

$$\mu(f_r) = \mu(f), \ f_r \in E_0, \ H(f_r) = H(f)$$

and $f_r \in E_h$. Clearly, $f_r \to f$ as $r \to 0^+$ in $\mathfrak{M}$. Therefore $V \cap E_h \neq \emptyset$. Lemma 15.3 is proved. □

Lemma 15.4. *Let $f \in E_0$, $h \in \mathfrak{A}$,*

$$\inf\{(\pi^f + h)(z) : z \in H(f)\} = \inf(\pi^f + h) \tag{15.10}$$

and let $\mathcal{V}$ be a neighborhood of f in $\mathfrak{M}$. Then $\mathcal{V} \cap E_h \neq \emptyset$.

Proof. There exists $z_0 \in R^n$ satisfying

$$z_0 \in H(f), \ (\pi^f + h)(z_0) = \inf(\pi^f + h). \tag{15.11}$$

For each $\lambda > 1$ define

$$f^{(\lambda)}(x, y) = \lambda f(x, y) + 2(\lambda - 1)a, \ x, y \in R^n \tag{15.12}$$

(a was used in A(ii)).

It is clear that for each $\lambda > 1$, $f^{(\lambda)} \in \tilde{\mathcal{M}}$ and $f^{(\lambda)} \to f$ as $\lambda \to 1^+$ in $\mathfrak{M}$. Thus there is $\lambda_0 > 1$ such that

$$f^{(\lambda)} \in V \text{ for all } \lambda \in (1, \lambda_0]. \tag{15.13}$$

Fix

$$\lambda \in (1, \lambda_0). \tag{15.14}$$

By (15.12), an a. c. function $v : [0, \infty) \to R^n$ is (f)-good if and only if it is $(f^{(\lambda)})$-good. This implies that $f^{(\lambda)}$ has (ATP) and

$$H(f^{(\lambda)}) = H(f),\ f^{(\lambda)} \in E_0. \tag{15.15}$$

By (15.12),

$$\pi^{f^{(\lambda)}}(z) = \lambda \pi^f(z) \forall z \in R^n. \tag{15.16}$$

If

$$\inf\{(\pi^{f^{(\lambda)}} + h)(z) :\ z \in H(f)\} > \inf(\pi^{(f^{(\lambda)}} + h),$$

then by Lemma 15.3, $V \cap E_h \neq \emptyset$.

Assume that

$$\inf\{(\pi^{f^{(\lambda)}} + h)(z) :\ z \in H(f)\} = \inf(\pi^{(f^{(\lambda)}} + h). \tag{15.17}$$

By (15.17), there exists $z_1 \in R^n$ satisfying

$$z_1 \in H(f),\ (\pi^{f^{(\lambda)}} + h)(z_1) = \inf(\pi^{f^{(\lambda)}} + h). \tag{15.18}$$

There exists a real-valued function $\xi \in C^\infty(R^n)$ such that the set $\{x \in R^n :\ \xi(x) \neq 0\}$ is bounded,

$$\xi(x) \leq 0 \text{ for all } x \in R^n, \tag{15.19}$$

$$0 > \xi(z_1),\ \xi(z) > \xi(z_1) \text{ for all } z \in R^n \setminus \{z_1\}. \tag{15.20}$$

For each $x = (x_1, \dots, x_n) \in R^n$ set $\nabla \xi(x) = (\partial \xi / \partial x_1(x), \dots, \partial \xi / \partial x_n(x))$ and denote by $\langle \cdot, \cdot \rangle$ the inner product in R^n.

For any $r \in (0,1)$ define a function $f_r^{(\lambda)} : R^n \times R^n \to R^1$ by

$$f_r^{(\lambda)}(x, y) = f^{(\lambda)}(x, y) - r\langle \nabla \xi(x), y \rangle,\ x, y \in R^n. \tag{15.21}$$

Arguing as in the proof of Lemma 10.3 of [36] we can show that there exists $r_0 \in (0,1)$ such that for every $r \in (0, r_0)$, $f_r^{(\lambda)} \in \tilde{\mathcal{M}}$. In view of (15.21), $f_r^{(\lambda)} \to f^{(\lambda)}$ in $\mathfrak{M}$ as $r \to 0^+$. Thus there exists $r \in (0, r_2)$ such that

$$f_r^{(\lambda)} \in V.$$

Arguing as in the proof of Lemma 10.3 of [36] we can show that $f_r^{(\lambda)} \in E_h$ and complete the proof of Lemma 15.4. □

Lemmas 15.3 and 15.4 imply the following result.

Lemma 15.5. *For every $h \in \mathfrak{A}$, the set E_h is an everywhere dense subset of $\mathfrak{M}$.*

16. Proofs of Theorems 6.5 and 6.8

Let $h \in \mathfrak{A}$ and let $\mathfrak{M}$ be one of the following spaces: $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\bar{\mathcal{M}}_q$, $q \geq 3$ is an integer. If $\mathfrak{M} = \bar{\mathcal{M}}_q$, where $q \geq 3$ is an integer, then we set $\tilde{\mathcal{M}} = \mathcal{M}_q$; if $\mathfrak{M} = \mathcal{L}_q$, where $q \in \{0, 1, 2\}$, then we set $\tilde{\mathcal{M}} = \mathcal{L}$. Denote by E_h the set of all $f \in \tilde{\mathcal{M}}$ which has (ATP) and for which there exists a unique (f)-perfect function $v_{f,h} : [0, \infty) \to R^n$ such that

$$(\pi^f + h)(v_{f,h}(0)) = \inf(\pi^f + h). \tag{16.1}$$

Let $f \in E_h$ and $k \geq 1$ be an integer. By Proposition 7.3, there exist an open neighborhood $\mathcal{U}(f)$ of f in $\mathcal{A}$ and a number $M(f) > 0$ such that the following property holds:

(i) for each $g \in \mathcal{U}(f)$ and each (g)-good function $x : [0, \infty) \to R^n$,

$$\limsup_{t \to \infty} |x(t)| < M(f).$$

By Theorem 14.1, there exist an open neighborhood $\mathcal{U}(f, h, k) \subset \mathcal{U}(f)$ of f in $\mathcal{A}$, an open neighborhood $\mathcal{V}(f, h, k)$ of h in $\mathfrak{A}$ and numbers $L(f, h, k) > k$, $\delta(f, h, k) \in (0, k^{-1})$ such that the following properties hold:

(ii) for each $g \in \mathcal{U}(f, h, k)$, each $\xi \in \mathcal{V}(f, h, k)$, each $T \geq L(f, h, k)$ and each a.c. function $u : [0, T] \to R^n$ which satisfies

$$|u(T)| \leq M(f) + k,$$

$$I^g(0, T, v) + \xi(v(0)) \leq \widehat{\sigma}(g, \xi, v(T), 0, T) + \delta(f, h, k)$$

we have $|u(t) - v_{f,h}(t)| \leq k^{-1}$ for all $t \in [0, k]$.

Assume that

$$g \in \mathcal{U}(f, h, k),\ \xi \in \mathcal{V}(f, h, k), \tag{16.2}$$

$z \in R^n$ satisfies

$$(\pi^g + \xi)(z) \leq \inf(\pi^g + \xi) + \delta(f, h, k) \tag{16.3}$$

and that $u : [0, \infty) \to R^n$ is an (g)-perfect function satisfying

$$u(0) = z. \tag{16.4}$$

Property (i) implies that there exists $T_0 > L(f, h, k)$ such that

$$|u(t)| < M(f) \text{ for all } t \geq T_0. \tag{16.5}$$

Fix

$$T \geq T_0 \tag{16.6}$$

and let an a. c. function $w : [0, T] \to R^n$ satisfy

$$w(T) = u(T). \tag{16.7}$$

In view of (16.5)-(16.7),

$$|u(T)| < M(f). \tag{16.8}$$

Since the function u is (g)-perfect it follows from (3.6), (3.7), (16.3), (16.4) and (16.7) that

$$I^g(0, T, w) + \xi(w(0))$$

$$= T\mu(g) + \Gamma^g(0,T,w) + \pi^g(w(0)) - \pi^g(w(T)) + \xi(w(0))$$
$$\geq T\mu(g) + \pi^g(w(0)) + \xi(w(0)) - \pi^g(u(T))$$
$$\geq T\mu(g) - \pi^g(u(T)) + (\pi^g + \xi)(u(0)) - \delta(f,h,k)$$
$$= I^g(0,T,u) + \xi(u(0)) - \delta(f,h,k).$$

Since the relation above holds for any function $w : [0,T] \to R^n$ satisfying (16.7) we conclude that

$$I^g(0,T,u) + \xi(u(0)) \leq \delta(f,h,k) + \widehat{\sigma}(g,\xi,u(T),0,T). \tag{16.9}$$

Property (ii), (16.2), (16.5), (16.6) and (16.9) imply that $|u(t) - v_{f,h}(t)| \leq k^{-1}$ for all $t \in [0,k]$. Thus we have shown that the following property holds:

(iii) for each $g \in \mathcal{U}(f,h,k)$, each $\xi \in \mathcal{V}(f,h,k)$ and each (g)-perfect function $u : [0,\infty) \to R^n$ satisfying $(\pi^g + \xi)(u(0)) \leq \inf(\pi^g + \xi) + \delta(f,h,k)$ we have $|u(t) - v_{f,h}(t)| \leq k^{-1}$ for all $t \in [0,k]$.

Completion of the proof of Theorem 6.5

Define

$$\mathcal{F} = \cap_{p=1}^{\infty} \cup \{\mathcal{U}(f,h,k) : \ f \in E_h, \ k \geq p\} \cap \mathfrak{M}. \tag{16.10}$$

In view of the construction and Lemma 15.5, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$.

Let $f \in \mathcal{F}$, $\epsilon, \tau_0, M > 0$. Assume that $v_1, v_2 : [0,\infty) \to R^n$ are (f)-perfect functions satisfying

$$(\pi^f + h)(v_i(0)) = \inf(\pi^f + h), \ i = 1,2 \tag{16.11}$$

which exists by Proposition 3.6.

Let a natural number p satisfy $p > \tau_0, M$ and $2p^{-1} < \epsilon$. By (16.10), for each integer $q \geq p$, there exist $f_q \in E_h$ and a natural number $k_q \geq q$ such that

$$f \in \mathcal{U}(f_q, h, k_q). \tag{16.12}$$

By (16.2), (16.11) and the property (iii), for $i = 1,2$, and all integers $q \geq p$,

$$|v_i(t) - v_{f_q,h}(t)| \leq k_q^{-1} \leq q^{-1}, \ t \in [0,k_q]. \tag{16.13}$$

This implies that $|v_1(t) - v_2(t)| \leq 2q^{-1}$ for all $t \in [0,q]$ and any integer $q \geq p$. Therefore $v_1(t) = v_2(t)$ for all $t \in [0,\infty)$ and there exists a unique (f)-perfect function $v_* : [0,\infty) \to R^n$ such that

$$(\pi^f + h)(v_*(0)) = \inf(\pi^f + h). \tag{16.14}$$

Clearly, $v_1 = v_2 = v_*$. By (16.3) and the equality above,

$$|v_*(t) - v_{f_p,h}(t)| \leq p^{-1}, \ t \in [0,p]. \tag{16.15}$$

Let

$$T_0 = L(f_p,h,k_p), \ g \in \mathcal{U}(f_p,h,k_p), \ \xi \in \mathcal{V}(f_p,h,k_p) \tag{16.16}$$

and $T \geq T_0$. Assume that an a. c. function $v : [0,T] \to R^n$ satisfies

$$|v(T)| \leq M,$$

$$I^g(0,T,v) + \xi(v(0)) \leq \widehat{\sigma}(g,\xi,v(T),0,T) + \delta(f_p,h,k_p). \tag{16.17}$$

By (16.6), (16.17) and the property (ii),

$$|v(t) - v_{f_p,h}(t)| \le k_p^{-1} \le p^{-1} \text{ for all } t \in [0, k_p]. \tag{16.18}$$

By (16.15) and (16.18), $|v(t) - v_*(t)| \le 2p^{-1} < \epsilon$ for all $t \in [0, p]$. Theorem 6.5 is proved.

Completion of the proof of Theorem 6.8

Define

$$\mathcal{F} = \cap_{p=1}^{\infty} \cup \{\mathcal{U}(f, h, k) \times \mathcal{V}(f, h, k) : \ h \in \mathfrak{A}, \ f \in E_h, \ k \ge p\} \cap (\mathfrak{M} \times \mathfrak{A}). \tag{16.19}$$

In view of the construction, Lemma 15.5 and (16.19), $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{M} \times \mathfrak{A}$.

Let $(f, h) \in \mathcal{F}$, $\epsilon, \tau_0, M > 0$. Assume that $v_1, v_2 : [0, \infty) \to R^n$ are (f)-perfect functions satisfying

$$(\pi^f + h)(v_i(0)) = \inf(\pi^f + h), \ i = 1, 2 \tag{16.20}$$

which exists by Proposition 3.6.

Let a natural number p satisfy $p > \tau_0, M$ and $2p^{-1} < \epsilon$. By (16.19), for each integer $q \ge p$, there exist $h_q \in \mathfrak{A}$, $f_q \in E_{h_q}$ and a natural number $k_q \ge q$ such that

$$(f, h) \in \mathcal{U}(f_q, h_q, k_q) \times \mathcal{V}(f_q, h_q, k_q). \tag{16.21}$$

By (16.20), (16.21) and the property (iii), for $i = 1, 2$, and all integers $q \ge p$,

$$|v_i(t) - v_{f_q,h_q}(t)| \le k_q^{-1} \le q^{-1}, \ t \in [0, k_q]. \tag{16.22}$$

This implies that $|v_1(t) - v_2(t)| \le 2q^{-1}$ for all $t \in [0, q]$ and any integer $q \ge p$. Therefore $v_1(t) = v_2(t)$ for all $t \in [0, \infty)$ and there exists a unique (f)-perfect function $v_* : [0, \infty) \to R^n$ such that

$$(\pi^f + h)(v_*(0)) = \inf(\pi^f + h).$$

Clearly, $v_1 = v_2 = v_*$. By (16.22) and the equality above,

$$|v_*(t) - v_{f_p,h_p}(t)| \le p^{-1}, \ t \in [0, p]. \tag{16.23}$$

Let

$$T_0 = L(f_p, h_p, k_p), \tag{16.24}$$

$$g \in \mathcal{U}(f_p, h_p, k_p), \ \xi \in \mathcal{V}(f_p, h_p, k_p) \tag{16.25}$$

and $T \ge T_0$. Assume that an a. c. function $v : [0, T] \to R^n$ satisfies

$$|v(T)| \le M,$$

$$I^g(0, T, v) + \xi(v(0)) \le \widehat{\sigma}(g, \xi, v(T), 0, T) + \delta(f_p, h_p, k_p). \tag{16.26}$$

By (16.24)-((16.26) and the property (ii),

$$|v(t) - v_{f_p,h_p}(t)| \le k_p^{-1} \le p^{-1} \text{ for all } t \in [0, k_p].$$

By the relation above and (16.23), $|v(t) - v_*(t)| \le 2p^{-1} < \epsilon$ for all $t \in [0, p]$. Theorem 6.8 is proved.

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