Applied Analysis and Optimization
Volume 1, Number 1, 2017, 1–17

Yokohama Publishers
ISSN 2189-1664 Online Journal

A STRONG CONVERGENCE THEOREM BY THE HYBRID METHOD FOR A NEW CLASS OF NONLINEAR OPERATORS IN A BANACH SPACE AND APPLICATIONS

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Abstract. In this paper, we first introduce a new class of nonlinear operators which covers strict pseudo-contractions and generalized hybrid mappings in Hilbert spaces and the sunny generalized nonexpansive retractions and the sunny generalized resolvents of maximal monotone operators in Banach spaces. Then, using the hybrid method, we prove a strong convergence theorem for the new class in a Banach space. Using the result, we obtain well-known and new strong convergence theorems in a Hilbert space and a Banach space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [6] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then

$$\|Ux - q\|^2 \leq \|x - q\|^2 + k\|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$. From this,

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we have that

$$2\langle x - Ux, x - q\rangle \geq (1 - k)\|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. We also know that there exists such a mapping in a Banach space. Let E be a smooth, strictly convex and reflexive Banach space and let E^* be the dual space of E. Let $B \subset E^* \times E$ be a maximal monotone operator. For each $r > 0$ and $x \in E$, Ibaraki and Takahashi [10] considered the set

$$J_r x := \{z \in E : x \in z + rBJ(z)\},$$

where J is the duality mapping on E. We know from [10] that $J_r x$ consists of one point. Such J_r is called the *sunny generalized resolvent* of B and is denoted by

$$J_r = (I + rBJ)^{-1}$$

2010 *Mathematics Subject Classification.* 47H05, 47H10.
Key words and phrases. Fixed point, generalized nonexpansive mapping, inverse strongly monotone mapping, hybrid method, maximal monotone operator, sunny generalized resolvent.
*The first author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.
†The second author was partially supported by the grant MOST 105-2115-M-039-002-MY3.

and the following property: for any $x \in E$ and $q \in (BJ)^{-1}0 = \{z \in E : 0 \in BJz\}$,

$$2\langle x - J_r x, JJ_r x - Jq\rangle \geq 0.$$

Then we get

$$2\langle x - J_r x, JJ_r x - Jx + Jx - Jq\rangle \geq 0$$

and hence

$$\begin{aligned} 2\langle x - J_r x, Jx - Jq\rangle &\geq 2\langle x - J_r x, Jx - JJ_r x\rangle \\ &= \phi(x, J_r x) + \phi(J_r x, x), \end{aligned} \tag{1.2}$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$.

On the other hand, in 2003, Nakajo and Takahashi [22] proved the following theorem by using the hybrid method:

Theorem 1.1 ([22]). *Let C be a nonempty, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence given by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection from H onto $C_n \cap Q_n$ and $\{\alpha_n\}$ is chosen so that $0 \leq \alpha_n \leq a < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

In this paper, motivated by (1.1) and (1.2), we first introduce a new class of nonlinear operators which covers strict pseudo-contractions and generalized hybrid mappings in Hilbert spaces and the sunny generalized nonexpansive retractions and the sunny generalized resolvents of maximal monotone operators in Banach spaces. Then, using the hybrid method, we prove a strong convergence theorem for the new class in a Banach space. Using the result, we obtain well-known and new strong convergence theorems in a Hilbert space and a Banach space.

2. Preliminaries

Let E be a Banach space with $\|\cdot\|$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We denote the value of x^* at x by $\langle x, x^*\rangle$. Then the duality mapping J on E defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. By the Hahn-Banach theorem, Jx is nonempty; see [28] for more details. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth.* It is said to be *Fréchet differntiable* if for any $x \in U$, the limit (2.1) is attained uniformly for all $y \in U$. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $x, y \in U$.

We also know the following properties (see [8, 25, 28, 29] for more details):

(1) $Jx \neq \emptyset$ for each $x \in E$.
(2) J is a monotone operator.
(3) If E is strictly convex, then J is one-to-one.
(4) If E is reflexive, then J is a mapping of E onto E^*.
(5) If E is smooth, then J is .
(6) E is uniformly convex if and only if E^* is uniformly smooth.
(7) If E is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded sets of E.

Let E be a smooth Banach space and let $\phi : E \times E \to [0, \infty)$ be the mapping defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $(x, y) \in E \times E$; see, for example, [1]. In a Hilbert space H, we have that

$$\phi(x, y) = \|x - y\|^2, \quad \forall x, y \in H.$$

We have that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

for all $x, y, z \in E$. We also have

$$2\langle x - y, Ju - Jv\rangle = \phi(x, v) + \phi(y, u) - \phi(x, u) - \phi(y, v) \tag{2.3}$$

for all $x, y, u, v \in E$. By the fact that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$ for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y.$$

Let E be a smooth, strictly convex and reflexive Banach space and let $\phi_* : E^* \times E^* \to [0, \infty)$ be the mapping defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for all $(x^*, y^*) \in E^* \times E^*$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx)$$

for all $x, y \in E$.

The following lemma was proved by Kamimura and Takahashi [13]:

Lemma 2.1 ([13])**.** *Let E be smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty and closed subset of a smooth Banach space E and let T be a mapping from C into E. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to E$ is called *generalized nonexpansive* [10] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y), \quad \forall (x, y) \in C \times F(T).$$

A point p in C is said to be a *generalized asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $\lim_{n\to\infty}(x_n - Tx_n) = 0$. We denote the set of generalized asymptotic fixed points of T by $\check{F}(T)$. Let D be a nonempty and closed subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E, \ \forall t \geq 0.$$

A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty and closed subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) from E onto D; see also [3,7,9,23,24] for sunny retractions.

Let E be a smooth, strictly convex and reflexive Banach space and let $B \subset E \times E^*$ be a set-valued mapping with the graph $G(B) = \{(x, x^*) : x^* \in Bx\}$ and the domain $D(B) = \{z \in E : Bz \neq \emptyset\}$. Then the mapping B is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall x, y \in D(B), \ x^* \in Bx, \ y^* \in By.$$

It is also said to be *maximal monotone* if B is monotone and its graph is not properly contained in the graph of any other monotone operator. It is known that if $B \subset E \times E^*$ is maximal monotone, then $B^{-1}0$ is closed and convex.

Let E be as above and let $B \subset E^* \times E$ be a maximal monotone operator. For each $r > 0$ and $x \in E$, consider the set

$$J_r x := \{z \in E : x \in z + rBJz\}.$$

Then $J_r x$ consists of one point; see [5,10,27]. Such J_r is called the *sunny generalized resolvent* of B and is denoted by

$$J_r = (I + rBJ)^{-1}.$$

The *Yosida approximation* of B is also denoted by $B_r = (I - J_r)/r$. It is shown in [10] that $(JJ_r x, B_r x) \in B$ for $x \in E$; see Ibaraki and Takahashi [9, 10] for more details.

Ibaraki and Takahashi [10] also proved some properties of J_r and $(BJ)^{-1}0$.

Proposition 2.2 ([10]). *Let E be a reflexive and strictly convex Banach space with a Fréchet differntiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the followings hold:*

(1) $D(J_r) = E, \quad \forall r > 0.$
(2) $(BJ)^{-1}0 = F(J_r), \quad \forall r > 0.$
(3) $(BJ)^{-1}0$ *is closed.*
(4) J_r *is generalized nonexpansive for each* $r > 0$.
(5) $\langle x - J_r, JJ_r x - Jq \rangle \geq 0, \quad \forall x \in E, \ q \in (BJ)^{-1}0.$

Using (2.3) and (5) in Proposion 2.2, we get that for all $x \in E$ and $y \in (BJ)^{-1}0$,

$$\phi(x, J_r x) + \phi(J_r x, y) \leq \phi(x, y). \tag{2.4}$$

They also proved the following lemmas:

Lemma 2.3 ([10]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.4 ([10]). *Let C be a nonempty and closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(1) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*
(2) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [15] proved the following results:

Theorem 2.5 ([15]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed subset of E. Then the following are equivalent:*

(1) *C is a sunny generalized nonexpansive retract of E;*
(2) *C is a generalized nonexpansive retract of E;*
(3) *JC is closed and convex.*

Proposition 2.6 ([15]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(1) *$z = Rx$;*
(2) *$\phi(x, z) = \min_{y \in C} \phi(x, y)$.*

In the case when a smooth, strictly convex and reflexive Banach space E is a Hilbert space H, we know that the sunny generalized nonexpansive retraction R_C from E onto C is the metric projection P_C from H onto C.

Let E be a smooth Banach space, let C be a nonempty and closed subset of E and let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. A mapping $T : C \to E$ with $F(T) \neq \emptyset$ is called (η, s)-generalized nonexpansive if, for any $x \in C$ and $q \in F(T)$,

$$2\langle x - Tx, Jx - Jq \rangle \geq (1 - \eta)\phi(Tx, x) + s\phi(x, Tx), \tag{2.5}$$

where J is the duality mapping on E. In particular, if $s = 0$ in (2.5), then the mapping T is as follows:

$$2\langle x - Tx, Jx - Jq \rangle \geq (1 - \eta)\phi(Tx, x)$$

for all $x \in C$ and $q \in F(T)$. Such $(\eta, 0)$-generalized nonexpansive mappings are important.

Examples.

(1) Let H be a Hilbert space, let C be a nonempty and closed subset of H and let k be a real number with $0 \leq k < 1$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is $(k, 0)$-generalized nonexpansive; see Introduction.

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.6}$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is $(0, 0)$-generalized nonexpansive. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (2.6), we have that

$$\alpha\|u - Ux\|^2 + (1-\alpha)\|u - Ux\|^2 \leq \beta\|u - x\|^2 + (1-\beta)\|u - x\|^2$$

and hence $\|Ux - u\|^2 \leq \|x - u\|^2$. From this, we have that

$$2\langle x - u, x - Ux\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is $(0, 0)$-generalized nonexpansive. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [16, 17] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [31] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [12].

(3) Let H be a Hilbert space and let C be a nonempty and closed subset of H. Let $\alpha > 0$. A mapping $A : C \to H$ is called α-inverse strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2 \tag{2.7}$$

for all $x, y \in C$; see [4, 21, 30, 33] for more details. Let $A : C \to H$ be an α-inverse strongly monotone mapping with $A^{-1}0 \neq \emptyset$. Then $1 - 2\alpha \in (-\infty, 1)$ and $I - A : C \to H$ is a $(1 - 2\alpha, 0)$-generalized nonexpansive mapping. In fact, setting $U = I - A$ and taking $y = z \in F(U) = A^{-1}0$ in (2.7), we have that

$$\langle x - z, x - Ux\rangle \geq \alpha\|x - Ux\|^2, \quad \forall x \in C,\ z \in F(U).$$

This implies that

$$2\langle x - z, x - Ux\rangle \geq (1 - (1 - 2\alpha))\|x - Ux\|^2, \quad \forall x \in C,\ z \in F(U)$$

and hence $U = I - A$ is $(1 - 2\alpha, 0)$-generalized nonexpansive.

(4) Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty, closed and convex subset of E. Let R_D be the sunny generalized nonexpansive retraction of E onto D. Then R_D is $(0, 1)$-generalized nonexpansive. In fact, since R_D is the sunny generalized nonexpansive retraction of E onto D, we have that, for any $x \in E$ and $q \in D$,

$$2\langle x - R_Dx, JR_Dx - Jq\rangle \geq 0.$$

Then we get

$$2\langle x - R_D x, JR_D x - Jx + Jx - Jq\rangle \geq 0$$

and hence

$$\begin{aligned} 2\langle x - R_D x, Jx - Jq\rangle &\geq 2\langle x - R_D x, Jx - JR_D x\rangle \\ &= \phi(x, R_D x) + \phi(R_D x, x). \end{aligned}$$

This means that R_C is $(0,1)$-generalized nonexpansive . Furthermore, since

$$\phi(x, R_D x) + \phi(R_D x, x) \geq \phi(R_D x, x),$$

R_D is also $(0,0)$-generalized nonexpansive.

(5) Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed subset of E. A mapping $U : C \to E$ is called generalized nonexpansive [10] if

$$\phi(Ux, z) \leq \phi(x, z) \quad \forall x \in C,\ z \in F(U). \tag{2.8}$$

See [2,19,20,26] for related mappings. If U is generalized nonexpansive and $F(U) \neq \emptyset$, then U is $(0,0)$-generalized nonexpansive. In fact, we have that, for $x \in C$ and $z \in F(U)$,

$$\phi(Ux, z) \leq \phi(x, z)$$

and hence

$$\phi(Ux, x) + \phi(x, z) + 2\langle Ux - x, Jx - Jz\rangle \leq \phi(x, z).$$

Therefore, we have that

$$\phi(Ux, x) \leq 2\langle x - Ux, Jx - Jz\rangle$$

for all $x \in C$ and $q \in F(U)$. This implies that U is $(0,0)$-generalized nonexpansive.

(6) Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. For each $r > 0$ and $x \in E$, consider the sunny generalized resolvent J_r [10], i.e.,

$$J_r = (I + rBJ)^{-1}.$$

Then the sunny generalized resolvent J_r with $B^{-1}0 \neq \emptyset$ is $(0,1)$-generalized nonexpansive. In fact, as in Introductin, we have that, for any $x \in E$ and $q \in (BJ)^{-1}0$,

$$2\langle x - J_r x, JJ_r x - Jq\rangle \geq 0.$$

Then we get

$$\begin{aligned} 2\langle x - J_r x, Jx - Jq\rangle &\geq 2\langle x - J_r x, Jx - JJ_r x\rangle \\ &= \phi(x, J_r x) + \phi(J_r x, x). \end{aligned}$$

This means that J_r is $(0,1)$-generalized nonexpansive. Furthermore, since

$$\phi(x, J_r x) + \phi(J_r x, x) \geq \phi(J_r x, x),$$

J_r is also $(0,0)$-generalized nonexpansive.

3. Main result

In this section, we prove a strong convergence theorem by the hybrid method for new nonlinear operators in a Banach space. Before proving the result, we prove the following lemma.

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$. If $T : C \to E$ is a $(\eta, 0)$-generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is closed and $JF(T)$ is closed and convex. In particular, if T is (η, s)-generalized nonexpansive, then $F(T)$ is closed and $JF(T)$ is closed and convex.*

Proof. We first prove that $F(T)$ is closed. Let $\{x_n\} \subset F(T)$ with $x_n \to x$. Since T is $(\eta, 0)$-generalized nonexpansive,

$$2\langle x - Tx, Jx - Jx_n\rangle \geq (1-\eta)\phi(Tx, x)$$

for all $n \in \mathbb{N}$. Since $x_n \to x$ and hence $x_n \rightharpoonup x$, we have $\phi(Tx, x) = 0$. Since E is stictly convex, it follows that $x \in F(T)$. This implies that $F(T)$ is closed.

We next show that $JF(T)$ is closed. Let $\{x_n^*\} \subset JF(T)$ such that $x_n^* \to x^*$ for some $x^* \in E^*$. Since JC is closed, there exist $x \in C$ and $\{x_n\} \subset F(T)$ such that $x^* = Jx$ and $x_n^* = Jx_n$ for all $n \in \mathbb{N}$. Since T is $(\eta, 0)$-generalized nonexpansive,

$$2\langle x - Tx, Jx - Jx_n\rangle \geq (1-\eta)\phi(Tx, x)$$

for all $n \in \mathbb{N}$. This implies that

$$2\langle x - Tx, x^* - x_n^*\rangle \quad \geq (1-\eta)\phi(Tx, x)$$

From $x_n^* \to x^*$, we have $\phi(Tx, x) = 0$ and hence $x^* = Jx \in JF(T)$. This implies that $JF(T)$ is closed.

We finally show that $JF(T)$ is convex. Let $x^*, y^* \in JF(T)$ and let $\alpha \in (0, 1)$ and $\beta = 1 - \alpha$. Then we have $x, y \in F(T)$ such that $x^* = Jx$ and $y^* = Jy$. Since T is $(\eta, 0)$-generalized nonexpansive,

$$2\langle x - Tx, Jx - Jp\rangle \geq (1-\eta)\phi(Tx, x), \quad \forall x \in C,\ p \in F(T).$$

Usiing (2.3), we have that

$$\phi(x, p) + \phi(Tx, x) - \phi(Tx, p) \geq (1-\eta)\phi(Tx, x)$$

and hence

$$\phi(x, p) + \eta\phi(Tx, x) \geq \phi(Tx, p).$$

Using this, we have that

$$\begin{aligned}
&\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
&\quad = \|TJ^{-1}(\alpha Jx + \beta Jy)\|^2 - 2\langle TJ^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&\qquad + \|J^{-1}(\alpha Jx + \beta Jy)\|^2 + \alpha\|x\|^2 + \beta\|y\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\quad = \alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) \\
&\qquad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\quad \leq \alpha(\phi(J^{-1}(\alpha Jx + \beta Jy), x) + \eta\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy))
\end{aligned}$$

$$
\begin{aligned}
&+ \beta(\phi(J^{-1}(\alpha Jx + \beta Jy), y) + \eta\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
&+ \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
={}& \alpha\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jx\rangle + \|x\|^2\big\} \\
&+ \beta\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jy\rangle + \|y\|^2\big\} \\
&+ \eta\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
&+ \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
={}& 2\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&+ \eta\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
={}& 2\|\alpha Jx + \beta Jy\|^2 - 2\|\alpha Jx + \beta Jy\|^2 \\
&+ \eta\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
={}& \eta\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)).
\end{aligned}
$$

Then we have

$$(1-\eta)\phi(TJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \le 0.$$

Since $1-\eta > 0$, we have $TJ^{-1}(\alpha Jx + \beta Jy) = J^{-1}(\alpha Jx + \beta Jy)$ and hence

$$\alpha Jx + \beta Jy \in JF(T).$$

Therefore, $JF(T)$ is convex. If T is (η, s)-generalized nonexpansive, then T is $(\eta, 0)$-generalized nonexpansive and hence $F(T)$ is closed and $JF(T)$ is closed and convex. This completes the proof. ☐

As a direct consequence of Theorem 2.5 and Lemma 3.1, we obtain the following result; see also [11].

Proposition 3.2. *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty and closed subset of E such that JC is closed and convex. Let $\eta \in (-\infty, 1)$. If $T : C \to E$ is a $(\eta, 0)$-generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let $\eta \in (-\infty, 1)$ and let $T : C \to E$ be a $(\eta, 0)$-generalized nonexpansive mapping such that $F(T) \neq \emptyset$ and assume that $\check{F}(T) = F(T)$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$
\begin{cases}
u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\
H_n = \{z \in C : 2\langle x_n - u_n, Jx_n - Jz\rangle \ge (1-\eta)\phi(u_n, x_n)\}, \\
W_n = \{z \in C : \langle x - x_n, Jz - Jx_n\rangle \le 0\}, \\
x_{n+1} = R_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\},
\end{cases}
$$

where $R_{H_n\cap W_n}$ is the sunny generalized nonexpansive retraction from E onto $H_n \cap W_n$ and $\{\alpha_n\} \subset [0, 1)$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction from E onto $F(T)$.

Proof. We first show that $F(T)$ is a sunny generalized nonexpansive retract of E. From Lemma 3.1, we have that $F(T)$ is closed and $JF(T)$ is closed and convex. Using Theorem 2.5, we have that $F(T)$ is a sunny generalized nonexpansive retract of E.

For each $n \in \mathbb{N} \cup \{0\}$, it is easy to see that H_n and W_n are closed since J is norm-to-weak* continuous. We also have that

$$JW_n = \{z^* \in C^* : \langle x - x_n, z^* - Jx_n\rangle \leq 0\}$$

and

$$JH_n = \{z^* \in C^* : 2\langle x_n - u_n, Jx_n - z^*\rangle \geq (1-\eta)\phi(u_n, x_n)\}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since E is smooth, strictly convex and reflexive, J is a single-valued bijection and hence

$$J(H_n \cap W_n) = JH_n \cap JW_n.$$

Thus JH_n, JW_n and $J(H_n \cap W_n)$ are closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

We show that $H_n \cap W_n$ is nonempty. We have that

$$\begin{aligned}
\phi(u_n, x_n) &= \|\alpha_n x_n + (1-\alpha_n)Tx_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)Tx_n, Jx_n\rangle + \|x_n\|^2 \\
&\leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tx_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)Tx_n, Jx_n\rangle + \|x_n\|^2 \\
&= \alpha_n\|x_n\|^2 - 2\alpha_n\langle x_n, Jx_n\rangle + \alpha_n\|x_n\|^2 \\
&\qquad + (1-\alpha_n)\|Tx_n\|^2 - (1-\alpha_n)\langle Tx_n, Jx_n\rangle + (1-\alpha_n)\|x_n\|^2 \\
&= \alpha_n\phi(x_n, x_n) + (1-\alpha_n)\phi(Tx_n, x_n) \\
&= (1-\alpha_n)\phi(Tx_n, x_n).
\end{aligned}$$

Let $w \in F(T)$. Since T is $(\eta, 0)$-generalized nonexpansive, we have that

$$\begin{aligned}
2\langle x_n - u_n, Jx_n - Jw\rangle &= 2(1-\alpha_n)\langle x_n - Tx_n, Jx_n - Jw\rangle \\
&\geq (1-\alpha_n)(1-\eta)\phi(Tx_n, x_n) \qquad (3.1)\\
&\geq (1-\eta)\phi(u_n, x_n).
\end{aligned}$$

Thus, we have $w \in H_n$ and hence $F(T) \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Next we show by induction that $F(T) \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. From $W_0 = C$, we have $F(T) \subset H_0 \cap W_0$. Suppose that $F(T) \subset H_k \cap W_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = R_{H_k \cap W_k}x$, we have that

$$\langle x - x_{k+1}, Jz - Jx_{k+1}\rangle \leq 0, \quad \forall z \in H_k \cap W_k$$

and hence

$$\langle x - x_{k+1}, Jw - Jx_{k+1}\rangle \leq 0, \quad \forall z \in F(T).$$

This implies $w \in W_{k+1}$. Hence $w \in H_{k+1} \cap W_{k+1}$. Thus we obtain $F(T) \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{x_n\}$ is well defined.

We show that $\lim_{n\to\infty} \phi(x, x_n)$ exists. Note that for each $n \in \mathbb{N} \cup \{0\}$, $x_n \in W_n$ and

$$\langle x - x_n, Jz - Jx_n\rangle \leq 0, \quad \forall z \in W_n.$$

By Lemma 2.4, we have $x_n = R_{W_n}x$. Using Lemma 2.4 again, we have

$$\phi(x, x_n) = \phi(x, R_{W_n}x) \leq \phi(x, z) - \phi(R_{W_n}x, z) \leq \phi(x, z), \quad \forall z \in F(T).$$

Thus $\{\phi(x, x_n)\}$ is bounded and hence $\{x_n\}$ is bounded. Since $x_n = R_{W_n}x$ and $x_{n+1} = R_{H_n \cap W_n}x \in H_n \cap W_n \subset W_n$, it follows from Proposition 2.6 that

$$\phi(x, x_n) \le \phi(x, x_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Thus $\{\phi(x, x_n)\}$ is nondecreasing and hence $\lim_{n\to\infty} \phi(x, x_n)$ exists.

We show that $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Consider

$$\begin{aligned}\phi(x_n, x_{n+1}) &= \phi(R_{W_n}x, x_{n+1}) \\ &\le \phi(x, x_{n+1}) - \phi(x, R_{W_n}x) \\ &= \phi(x, x_{n+1}) - \phi(x, x_n).\end{aligned}$$

Therefore, $\lim_{n\to\infty} \phi(x_n, x_{n+1}) = 0$. From Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.2}$$

Since E is uniformly smooth, we have that

$$\lim_{n\to\infty} \|Jx_n - Jx_{n+1}\| = 0. \tag{3.3}$$

From $x_{n+1} = R_{H_n \cap W_n}x \in H_n$, we also have

$$2\langle x_n - u_n, Jx_n - Jx_{n+1}\rangle \ge (1-\eta)\phi(u_n, x_n). \tag{3.4}$$

Furthermore, we claim that $\{x_n - u_n\}$ is bounded. For showing that $\{x_n - u_n\}$ is bounded, we may prove that $\{u_n\}$ is bounded. Since

$$2\langle x_n - u_n, Jx_n - Jz\rangle \ge (1-\eta)\phi(u_n, x_n)$$

for $z \in F(T)$, we have from (2.3) that

$$\phi(x_n, z) + \phi(u_n, x_n) - \phi(u_n, z) \ge (1-\eta)\phi(u_n, x_n)$$

and hence

$$\eta\phi(u_n, x_n) + \phi(x_n, z) \ge \phi(u_n, z).$$

In the case of $\eta \le 0$, we have $\phi(x_n, z) \ge \phi(u_n, z)$. So, we have that, for $z \in F(T)$,

$$(\|u_n\| - \|z\|)^2 \le \phi(u_n, z) \le \phi(x_n, z) \le (\|x_n\| + \|z\|)^2.$$

Using this, we have that $|\|u_n\| - \|z\|| \le \|x_n\| + \|z\|$ and hence

$$\|u_n\| \le \|x_n\| + \|z\| + \|z\| \le \|x_n\| + 2\|z\|.$$

Then, we have that $\{u_n\}$ is bounded. In the case of η withh $0 < \eta < 1$, we have

$$\eta\phi(u_n, x_n) + \phi(x_n, z) \ge \phi(u_n, z).$$

So, we have that, for $z \in F(T)$,

$$\begin{aligned}(\|u_n\| - \|z\|)^2 &\le \phi(u_n, z) \\ &\le \phi(u_n, z) + \eta\phi(u_n, x_n) \\ &\le (\|u_n\| + \|z\|)^2 + \eta(\|u_n\| + \|x_n\|)^2 \\ &\le \big(\|u_n\| + \|z\| + \sqrt{\eta}(\|u_n\| + \|x_n\|)\big)^2.\end{aligned}$$

From this, we have that

$$|\|u_n\| - \|z\|| \le \|x_n\| + \|z\| + \sqrt{\eta}(\|u_n\| + \|x_n\|)$$

and hence

$$(1-\sqrt{\eta})\|u_n\| \le (1+\sqrt{\eta})\|x_n\| + 2\|z\|.$$

Then, we have that

$$\|u_n\| \le \frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|.$$

This implies that $\{u_n\}$ is bounded. Since $\|Jx_n - Jx_{n+1}\| \to 0$ from (3.3) and $\{x_n - u_n\}$ is bounded, we get from (3.4) that

$$\lim_{n\to\infty} \phi(u_n, x_n) = 0. \tag{3.5}$$

Therefore, we get from Lemma 2.1 that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.6}$$

Since $u_n = \alpha_n x_n + (1-\alpha_n)Tx_n$, we have that

$$\|x_n - u_n\| = (1-\alpha_n)\|x_n - Tx_n\|.$$

We have from (3.6) and $\limsup_{n\to\infty} \alpha_n < 1$ that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.7}$$

Since $\{Jx_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $Jx_{n_i} \rightharpoonup z^*$. Since $J(C)$ is closed and convex, we have $z^* \in J(C)$ and hence $J^{-1}z^* \in C$. From (3.7), we have that $J^{-1}z^* \in \check{F}(T)$. Putting $z = J^{-1}z^*$, we have $z \in \check{F}(T)$.

We next show that $z = R_{F(T)}x$. Let $u = R_{F(T)}x$. From $x_{n+1} = R_{H_n \cap W_n}x$ and $u \in F(T) \subset H_n \cap W_n$, we have

$$\phi(x, x_{n+1}) \le \phi(x, u).$$

From $Jx_{n_i} \rightharpoonup Jz$, we have

$$\begin{aligned}
\phi(x, z) &= \|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2 \\
&\le \liminf_{i\to\infty} \left(\|x\|^2 - 2\langle x, Jx_{n_i}\rangle + \|x_{n_i}\|^2\right) \\
&= \liminf_{i\to\infty} \phi(x, x_{n_i}) \\
&\le \limsup_{i\to\infty} \phi(x, x_{n_i}) \\
&\le \phi(x, u).
\end{aligned}$$

From the definition of u, we have $u = z$. Thus we obtain $z^* = Jz = Ju$ and hence $Jx_n \rightharpoonup z^* = Jz = Ju$.

We finally show that $x_n \to z$. From (2.2), we have that

$$\phi(z, x_n) = \phi(z, x) + \phi(x, x_n) + 2\langle z - x, Jx - Jx_n\rangle, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Since $x_n = R_{W_n}x$ and $z \in F(T) \subset W_n$, $\phi(x, x_n) \le \phi(x, z)$ and hence

$$\begin{aligned}
\limsup_{n\to\infty} \phi(z, x_n) &= \limsup_{n\to\infty} \{\phi(z, x) + \phi(x, x_n) + 2\langle z - x, Jx - Jx_n\rangle\} \\
&\le \limsup_{n\to\infty} \{\phi(z, x) + \phi(x, z) + 2\langle z - x, Jx - Jx_n\rangle\} \\
&= \phi(z, x) + \phi(x, z) + 2\langle z - x, Jx - Jz\rangle \\
&= \phi(z, z) = 0.
\end{aligned}$$

Thus $\lim_{n\to\infty} \phi(z, x_n) = 0$ and hence $\lim_{n\to\infty} \|z - x_n\| = 0$. This complete the proof. □

4. Applications

In this section, using Theorem 3.3, we can obtain well-known and new strong convergence theorems in a Hilbert space and a Banach space. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds. In the case when a smooth, strictly convex and reflexive Banach space E is a Hilbert space, we know that $\check{F}(U) = F(U)$ is equivalent to the demiclosedness of U.

We know the following lemmas obtained by Marino and Xu [18] and Kocourek, Takahashi and Yao [14]; see also [32, 34].

Lemma 4.1 ([18, 32]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \le k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Lemma 4.2 ([14, 34]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

The followng is a strong convergence theorem for strict pseudo-contractions in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in [0, 1)$ and let T be a k-strict pseudo-contraction of C into H. Assume that $F(T) \neq \emptyset$. Let $x_0 = x \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ H_n = \{z \in C : 2\langle x_n - u_n, x_n - z\rangle \ge (1 - k)\|u_n - x_n\|^2\}, \\ W_n = \{z \in C : \langle x - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $P_{H_n\cap W_n}$ is the metric projection from H onto $H_n \cap W_n$ and $\{\alpha_n\} \subset [0, 1)$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.

Proof. Since T is a k-strict pseudo-contraction of C into H such that $F(T) \neq \emptyset$, from (1) in Examples, T is $(k, 0)$-generalized nonexpansive. Furthermore, from Lemma 4.1, T is demiclosed. Therefore, we have the desired result from Theorem 3.3. □

The followng is a strong convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a generalized hybrid mapping of C into H. Assume* *that $F(T) \neq \emptyset$. Let $x_0 = x \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ H_n = \{z \in C : \|u_n - z\| \leq \|x_n - z\|\}, \\ W_n = \{z \in C : \langle x - x_n, z - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $P_{H_n \cap W_n}$ is the metric projection from H onto $H_n \cap W_n$ and $\{\alpha_n\} \subset [0,1)$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.

Proof. Since T is a generalized hybrid mapping of C into H such that $F(T) \neq \emptyset$, from (2) in Examples, T is $(0,0)$-generalized nonexpansive. Furthermore, from Lemma 4.2, T is demiclosed. Since T is $(0,0)$-generalized nonexpansive, we also have that the inequality $2\langle x_n - u_n, Jx_n - Jz\rangle \geq (1-\eta)\phi(u_n, x_n)$ in Theorem 3.3 is as

$$2\langle x_n - u_n, x_n - z\rangle \geq \|u_n - x_n\|^2.$$

Using (2.3), we have that $\|u_n - z\|^2 \leq \|x_n - z\|^2$ and hence $\|u_n - z\| \leq \|x_n - z\|$. Therefore, we have the desired result from Theorem 3.3. □

The followng is a strong convergence theorem for inverse strongly monotone mappings in a Hilbert space.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\mu \in (0,\infty)$. Let B be a μ-inverse strongly monotone mapping of C into H and let $T = I - B$. Assume that $B^{-1}0 = \{z \in C : Bz = 0\} \neq \emptyset$. Let $x_0 = x \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ H_n = \{z \in C : \langle x_n - u_n, x_n - z\rangle \geq \mu\|u_n - x_n\|^2\}, \\ W_n = \{z \in C : \langle x - x_n, z - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $P_{H_n \cap W_n}$ is the metric projection from H onto $H_n \cap W_n$ and $\{\alpha_n\} \subset [0,1)$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.

Proof. Since B is a μ-inverse strongly monotone mapping of C into H such that $B^{-1}0 \neq \emptyset$, from (3) in Examples, $T = I - B$ is $(1-2\mu, 0)$-generalized nonexpansive. So, the inequality $2\langle x_n - u_n, Jx_n - Jz\rangle \geq (1-\eta)\phi(u_n, x_n)$ in Theorem 3.3 is as

$$2\langle x_n - u_n, x_n - z\rangle \geq (1 - (1-2\mu))\|u_n - x_n\|^2$$

and hence $\langle x_n - u_n, x_n - z\rangle \geq \mu\|u_n - x_n\|^2$. Furthermore, T is demiclosed. In fact, since

$$I - 2\mu B = I - 2\mu(I - T) = (1-2\mu)I + 2\mu T$$

and $I - 2\mu B$ is nonexpansive from [30], we have that $(1-2\alpha)I + 2\alpha T$ is nonexpansive. So, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, we have that

$$x_n - ((1-2\mu)I + 2\mu T)x_n = 2\mu(I - T)x_n \to 0.$$

Since $(1-2\mu)I + 2\mu T$ is nonexpansive, it is demiclosed; see [30]. Then, we have $z \in F((1-2\mu)I + 2\mu T) = F(T)$. This implies that T is demiclosed. Therefore, we have the desired result from Theorem 3.3. □

Theorem 4.6. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let $T : C \to E$ be a generalized nonexpansive mapping such that $F(T) \neq \emptyset$ and assume that $\check{F}(T) = F(T)$. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ H_n = \{z \in C : \phi(u_n, z) \le \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle x - x_n, Jz - Jx_n \rangle \le 0\}, \\ x_{n+1} = R_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1)$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction from E onto $F(T)$.

Proof. Since $T : C \to E$ is generalized nonexpansive, from (5) in Examples, it is $(0,0)$-generalized nonexpansive. So, $2\langle x_n - u_n, Jx_n - Jz \rangle \ge (1-\eta)\phi(u_n, x_n)$ in Theorem 3.3 is as

$$2\langle x_n - u_n, Jx_n - Jz \rangle \ge \phi(u_n, x_n).$$

Using (2.3), we have that $\phi(u_n, z) \le \phi(x_n, z)$. Therefore, we have the desired result from Theorem 3.3. □

Corollary 4.7. *Let E be a uniformly convex and uniformly smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with $(BJ)^{-1}0 \neq \emptyset$ and let $J_r = (I + rBJ)^{-1}$ for all $r > 0$. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)J_r x_n, \\ H_n = \{z \in E : \phi(u_n, z) \le \phi(x_n, z)\}, \\ W_n = \{z \in E : \langle x - x_n, Jz - Jx_n \rangle \le 0\}, \\ x_{n+1} = R_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1)$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}x$, where $R_{(BJ)^{-1}0}$ is the sunny generalized nonexpansive retraction from E onto $(BJ)^{-1}0$.

Proof. Since $J_r : E \to E$ is $(0,0)$-generalized nonexpansive. As in the proof of Theorem 4.6, the inequality $2\langle x_n - u_n, Jx_n - Jz \rangle \ge (1-\eta)\phi(u_n, x_n)$ in Theorem 3.3 is as

$$\phi(u_n, z) \le \phi(x_n, z).$$

Furthermore, we have $\check{F}(J_r) = F(J_r)$. In fact, assume that $Jx_n \rightharpoonup Jp$ and $x_n - J_r x_n \to 0$. Since E is uniformly smooth, we have that $\|Jx_n - JJ_r x_n\| \to 0$. It is clear that $JJ_r x_n \rightharpoonup Jp$. Since J_r is the sunny generalized resolvent of B, we have that

$$\langle x_n - J_r x_n - (p - J_r p), JJ_r x_n - JJ_r p \rangle \ge 0.$$

Therefore, $\langle p - J_r p, Jp - JJ_r p \rangle \le 0$. This implies that

$$\phi(p, J_r p) + \phi(J_r p, p) \le 0$$

and hence $p = J_r p$. Then, we have $\check{F}(J_r) = F(J_r)$. Therefore, we have the desired result from Theorem 3.3. □

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Manuscript received 2 March 2017
revised 24 March 2017

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