Pure and Applied Functional Analysis
Volume 5, Number 2, 2020, 427–455

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ESSENTIAL SPECTRUM OF PERIODIC MEDIUM WITH SPARSELY PLACED FOREIGN INCLUSIONS

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Abstract. We study the essential spectrum of a formally self-adjoint system of partial differential equations, for example the elasticity system, in periodic domains with non-periodic perturbations. The perturbation is realized as a sparse distribution of identical foreign cells in a periodic medium in $\mathbb{R}^d$, $d \geq 2$. It is shown that the essential spectrum consists of the essential spectrum, with band-gap structure, of the corresponding problem in the purely periodic medium and of the discrete spectrum of the model problem, where the periodicity is broken by one foreign cell only. The increment of the essential spectrum, caused by the perturbation, may occur either inside the spectral gaps, or below the spectrum of the unperturbed problem. We also discuss generalizations and open questions.

1. Introduction

1.1. **Motivation and aim of the paper.** Composite heterogeneous media and structures in macro- and mesoscales are used in many types of devices produced by the modern engineering. Mathematically, purely periodic composites can be and have been investigated by using the Floquet-Bloch-Gelfand (FBG) -theory and basic methods of the spectral theory of self-adjoint semibounded operators. In contrast to homogeneous media, the spectrum of a periodic medium has band-gap structure with alternating passing zones (bands) and stopping zones (gaps), which allow or prevent, respectively, waves with the frequencies in the zone to propagate. The passing zones may overlap and there may exist no open gaps, in which case waves with any frequency above the cut-off value of the essential spectrum can propagate.

Nevertheless, in the reality it is almost impossible to produce absolutely periodic composites; all materials with periodic structure have some defects distributed within them. It should be emphasized that one can also insert foreign inclusions on purpose in order to create composites with desired properties. There does not exist much literature on the spectra of periodic media with non-periodic perturbations, and the main goal of this paper is to compensate this shortcoming. We will in particular show that even a sparse distribution of foreign inclusions in periodic media can cause increments to the essential spectrum. We formulate this main result of our paper as Theorem 2.3, and it concerns a general class of formally self-adjoint

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2010 *Mathematics Subject Classification.* 35J57, 35P99.

Key words and phrases. Elliptic boundary problem, periodic domain, essential spectrum.

*The first named author was partially supported by the grant 18-01-00325 of the Russian Foundation of Basic Research and the grant 308829 of the Academy of Finland.

elliptic spectral second order boundary value problems (1.20), (1.21) in domains $\Pi^\sharp$, (1.19).

The above-mentioned peculiarity of wave phenomena is realized mathematically as the emergence of new points of the essential spectrum, either inside the spectral gaps or below the spectrum of the purely periodic medium. These points come from the discrete spectrum of the model problem, where only one cell is changed into a foreign one.

As for the structure of this paper, the exact statements of the purely periodic, perturbed and model problems will be given in Sections 1.2–1.3, and we in particular specify the meaning of "sparcely distributed" foreign cells. Besides, in Section 1.4 we present examples of problems in mathematical physics, to which our results apply directly, while in Section 4 we will discuss possible generalizations and modifications of the main result yielding more examples. Section 2 contains descriptions of the spectra of the introduced problems as well as the formulation of our main result in Theorem 2.3. Its proof, divided into several steps, is presented in Section 3.

Sparse perturbations of the Schrödinger equation with decaying potentials were studied in the papers [15, 16, 23, 24]. In these papers there were found isolated points of the essential spectrum, which are below the cut-off value of the spectrum of the equation with the original potential. Our technique allows to study the spectrum of the Schrödinger equation with non-decaying periodic potentials with sparsely placed local perturbations; it can yield points of the essential spectrum inside the spectral gaps.

In the paper [36] we considered one-side directed periodic quantum waveguides (the spectral Dirichlet-Laplace problem) with local perturbations sparsely distributed along the waveguide axis. The final theorem in [36] is similar to our Theorem 2.3, but the technique in the reference is completely based on general comprehensive results in [25], [30, Ch. 3,5] about solvability of elliptic problems in periodic quasi-cylinders and asymptotics of their solutions at infinity. Such theory is not yet known in domains which are periodic in many dimensions. Serious problems for the present analysis are caused by the necessity to treat the variational formulation of systems of differential equations and mixed boundary value problems when the coefficients and boundaries are not assumed smooth[1] ; also the band-gap spectrum of the unperturbed problem brings additional difficulties. The most technical issue of our paper, Theorem 3.4 on the exponential decay of the solutions of the model problem, will be proved by using several new tricks, namely by verifying the Fredholm property and deriving a priori estimates in weighted Sobolev spaces without directly using the FBG-transform. It should be mentioned that we employ in parallel two dissimilar operator realizations of the variational problems, the spectra of which have a simple relationship. These operators are used to verify different particular properties of the perturbed problem.

[1]This is a direct requirement of the main application to the elasticity, since the elastic moduli of composites are usually only piecewise continuous and those associated with fractures creating micro-cracks are even less smooth.

1.2. **Formulation of the periodic problem.** Let ϖ be the periodicity cell, which is an open subset of the unit cube

$$\square = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : |x_j| < 1/2,\ j = 1, \ldots, d\}. \tag{1.1}$$

We denote by Π the interior of the union

$$\overline{\Pi} = \bigcup_{\alpha \in \mathbb{Z}^d} \overline{\varpi(\alpha)}, \tag{1.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $\mathbb{Z}$ is the set of integers and

$$\varpi(\alpha) = \{x : x - \alpha \in \varpi\}. \tag{1.3}$$

Let Π be a domain, in particular a connected set, which has a $(d-1)$-dimensional Lipschitz boundary $\partial\Pi$. In order to properly formulate the boundary value problem

$$L(x, \nabla)u(x) = \lambda M(x)u(x), \quad x \in \Pi, \tag{1.4}$$

$$B(x, \nabla)u(x) = 0, \qquad x \in \partial\Pi, \tag{1.5}$$

we assume for a moment that the boundary and other data are smooth, but after going over to the weak statement (1.14) we will return to the Lipschitz case. The differential operator of the system (1.4) is given by

$$L(x, \nabla) = \overline{D(-\nabla)}^{\top} A(x) D(\nabla), \tag{1.6}$$

where ∇ is the gradient, A and M are Hermitian positive matrices of size $N \times N$ and $n \times n$, respectively, and $D(\nabla)$ is an $(N \times n)$-matrix of first order differential operators with constant complex coefficients so that $\overline{D(-\nabla)}^{\top}$ is the formal adjoint of $D(\nabla)$; the transposition of matrices is denoted by $\top$. We require that D is *algebraically complete* [37], that is, there exists a number $\varrho_D \in \mathbb{N} = \{1, 2, 3, \ldots\}$ such that, for for any row p of homogeneous polynomials $p_1, \ldots, p_m$ of degree $\varrho \le \varrho_D$, one can find a row $q = (q_1, \ldots, q_N)$ of polynomials satisfying the relation

$$p(\xi) = q(\xi) D(\xi) \quad \forall\ \xi \in \mathbb{R}^d. \tag{1.7}$$

In other words, $p(\xi)$ can be divided by $D(\xi)$. Property (1.7) assures that L is a *formally positive operator* [37, § 3.7.4], namely, there holds the Korn inequality

$$\|u; H^1(\varpi)\|^2 \le c\big(a(u, u; \varpi) + \|u; L^2(\varpi)\|\big)^2, \tag{1.8}$$

where $H^1(\varpi)$ and $L^2(\varpi)$ are the Sobolev and Lebesgue spaces, the coefficient $c > 0$ depends on ϖ and D, A, but not on $u = (u_1, \ldots, u_n)^{\top} \in H^1(\varpi)^n$ and a is the Hermitian positive form

$$a(u, v; \varpi) = \big(AD(\nabla)u, D(\nabla)v\big)_{\varpi}. \tag{1.9}$$

For all vector functions u, $v \in H^2_{\mathrm{per}}(\varpi)^n$, which are 1-periodic in the variables $x_1, \ldots, x_d$, there holds the Green formula

$$a(u, v; \varpi) = (Lu, v)_{\varpi} + (Nu, v)_{\upsilon} \tag{1.10}$$

where $\upsilon = \partial\varpi \cap \square$ is the "interior" boundary of the cell, possibly the empty set, and

$$N(x, \nabla) = \overline{D(\nu(x))}^{\top} A(x) D(\nabla) \tag{1.11}$$

with the unit outward normal vector ν. The boundary condition operator is given by

$$B(x,\nabla) = \big(\mathbb{I}_n - P(x)\big)N(x,\nabla) + P(x), \tag{1.12}$$

where $\mathbb{I}_n$ and $\mathbb{O}_n$ are the unit and null matrices of size $n\times n$ and $P(x)$ is an orthogonal projection in $\mathbb{C}^n$, which may depend continuously on x belonging to the compact set υ. Thus, (1.5) coincides with the Dirichlet condition in the case $P = \mathbb{I}_n$ and the Neumann one, if $P = \mathbb{O}_n$.

For a vector function $u \in H^2(\Pi)$ satisfying the boundary condition (1.5) and a vector function $v \in \mathcal{H}(\Pi)$, where

$$\mathcal{H}(\Pi) = \{v \in H^1(\Pi)^n \,:\, Pv = 0 \text{ on } \partial\Pi\}, \tag{1.13}$$

the last scalar product in (1.11) vanishes. Hence, formulas (1.10)-(1.13) yield the variational formulation of the problem (1.4), (1.5): find $u \in \mathcal{H}(\Pi)$ and $\lambda \in \mathbb{C}$ such that

$$a(u,v;\Pi) = \lambda(Mu,v)_\Pi \quad \forall\, v \in \mathcal{H}(\Pi). \tag{1.14}$$

Notice that in the variational problem (1.14) it suffices that the matrix functions A, M and P are bounded and measurable (instead of smooth). We extend them 1-periodically from ϖ and υ to Π and $\partial\Pi$, respectively, and require that

$$\begin{aligned} &C_A|\eta|^2 \ge \overline{\eta}^\top A(x)\eta \ge c_A|\eta|^2 \quad \forall\, \eta \in \mathbb{C}^N,\\ &C_M|\zeta|^2 \ge \overline{\zeta}^\top M(x)\zeta \ge c_M|\zeta|^2 \quad \forall\, \zeta \in \mathbb{C}^n, \end{aligned} \tag{1.15}$$

where C_A, c_A, C_M, c_M are positive constants independent of $x \in \varpi$, η and ζ. Also, the normal vector ν is defined almost everywhere on the Lipschitz surfaces υ and $\partial\Pi$.

In Section 2.1 we will give the equivalent operator formulation of the problem (1.14) and define its spectrum σ properly.

1.3. **Formulation of the perturbed problem.** Let $\varpi^\bullet \subset \square$ be a foreign cell such that

$$\Pi^\circ = (\Pi \setminus \varpi) \cup \varpi^\bullet \tag{1.16}$$

is still a domain with Lipschitz boundary (in addition we assume about the geometry of $\varpi^\bullet$ that the domain (1.19), below, will be Lipschitz). We set

$$A^\circ = A \text{ in } \Pi\setminus\varpi, \quad A^\circ = A^\bullet \text{ in } \varpi^\bullet, \tag{1.17}$$

where $A^\bullet$ is a foreign $N\times N$-matrix with the same general properties as A. We define the matrices M° and P° analogously to (1.17) by using the original and foreign matrices M, P and $M^\bullet$, $P^\bullet$, respectively, where the latter also have the qualities described after (1.6). We apply self-evident changes to the notation in (1.14) and (1.13) (see also the beginning of Section 2.2, below) and pose the problem

$$a^\circ(u^\circ,v^\circ;\Pi^\circ) = \lambda^\circ(M^\circ u^\circ,v^\circ)_{\Pi^\circ} \quad \forall\, v^\circ \in \mathcal{H}(\Pi^\circ), \tag{1.18}$$

the spectrum of which will be studied in Section 2.2. Of course, in order to avoid trivialities, we assume that at least one of $\varpi^\bullet$ $A^\bullet$, $M^\bullet$, or $P^\bullet$, differs from the corresponding original objects.

Let $\{\alpha^k\}_{k\in\mathbb{N}}$ be a sequence of multi-indices in $\mathbb{Z}^d$ such that the numbers $|\alpha^k| = |\alpha_1^k| + \ldots + |\alpha_d^k|$ form a monotonely increasing, unbounded sequence. Replacing the cells (1.3) with indices α^k by the foreign cells

$$\varpi^\bullet(\alpha) = \{x \,:\, x - \alpha \in \varpi^\bullet\}$$

we obtain the modified domain

$$\Pi^\sharp = \Big(\Pi \setminus \bigcup_{k\in\mathbb{N}} \varpi(\alpha^k)\Big) \cup \bigcup_{k\in\mathbb{N}} \varpi^\bullet(\alpha^k). \tag{1.19}$$

Similarly to (1.17) we define the matrices $A^\sharp$, $M^\sharp$ on $\Pi^\sharp$ and $\Theta^\sharp$ on $\partial\Pi^\sharp$ by substituting the original matrices in the selected cells $\varpi(\alpha^k)$, $k \in \mathbb{N}$. Our principal object of investigation is the spectral boundary value problem

$$L^\sharp(x,\nabla)u(x) = \lambda M^\sharp(x)u(x), \quad x \in \Pi^\sharp, \tag{1.20}$$

$$B^\sharp(x,\nabla)u(x) = 0, \qquad\qquad x \in \partial\Pi^\sharp, \tag{1.21}$$

where the differential operators $L^\sharp$ and $B^\sharp$ are defined as in (1.6) and (1.12) by changing $A \mapsto A^\sharp$ and $P \mapsto P^\sharp$. By $\sigma^\sharp$ we understand the spectrum of the variational form of the problem (1.20), (1.21), namely

$$a^\sharp(u^\sharp, v^\sharp; \Pi^\sharp) = \lambda^\sharp(M^\sharp u^\sharp, v^\sharp)_{\Pi^\sharp} \quad \forall\ v^\sharp \in \mathcal{H}(\Pi^\sharp), \tag{1.22}$$

where $a^\sharp$ and $\mathcal{H}(\Pi^\sharp)$ are obtained from (1.9) and (1.13) by using $A^\sharp$ and $P^\sharp$ instead of A and P.

Remark 1.1. If $\{\alpha^k\}_{k\in\mathbb{N}} = \mathbb{Z}^d$, then we again obtain a 1-periodic medium the spectrum of which can be studied by the FBG-theory as will be outlined in Section 2.1. Many other choices of the sequence $\{\alpha^k\}_{k\in\mathbb{N}}$ lead to purely periodic media. For example the one corresponding to the chessboard distribution of the foreign cells only means the doubling of the length of the period.

To describe the *sparse distribution* of foreign cells we denote for every $p \in \mathbb{N}$ by $L_p > 0$ the largest natural number such that for the cube

$$\boxdot^p = \{x \,:\, |x_j - \alpha_j^p| < L_p + 1/2, j = 1, \ldots, d\} \tag{1.23}$$

there holds $\boxdot^p \cap \varpi^\bullet(\alpha^k) = \varnothing$ for all $k \neq p$. We now make the principal geometric assumption of this paper by requiring that

$$\lim_{k\to+\infty} L_k = +\infty. \tag{1.24}$$

Remark 1.2. Suppose that condition (1.24) holds with $\tilde{L}_k$ for some sequence $\{\widetilde{\alpha}^k\}_{k\in\mathbb{N}} \subset \mathbb{Z}^d$. If a new sequence $\{\alpha^p\}_{p\in\mathbb{N}}$ is defined by $\alpha^{2k-1} = \widetilde{\alpha}^{2k-1}$ and $\alpha^{2k} = \alpha^{2k-1} + (1, 0, \ldots, 0)$ for all $k \in \mathbb{N}$, then $L_p = 0$ for every $p \in \mathbb{N}$ so that (1.24) does certainly not hold. However, we perform the change of coordinates $x \mapsto x' = (x_1/2, x_2, \ldots, x_d)$ and redefine the matrix differential operator $D(\nabla)$ accordingly, but still regard the obtained matrices and domain Π' as 1-periodic in all directions. In this way we join the cells $\varpi'(\alpha^{2k-1})$ and $\varpi'(\alpha^{2k})$, which allows us to redetermine L'_p and see that condition (1.24) holds. Apparently, this idea can be modified and generalized in many ways; see also Sections 3.3 and 4.1.

Remark 1.3. Let $\alpha^k = (kN, 0, \dots, 0)$ for $k \in \mathbb{N}$ and some fixed $N \in \mathbb{N}$. Then, $L_k = N - 1$ and condition (1.24) fails. However, the foreign inclusions $\varpi^\bullet(\alpha^k)$, $k \in \mathbb{N}$, form a so called *open waveguide*, and the corresponding problem (1.22) has been thoroughly investigated in [5].

1.4. **Concrete spectral problems in mathematical physics.** 1°. *Scalar case.* Let $n = 1$, $N = d$ and $D(\nabla) = \nabla$. Then, (1.6) is a scalar elliptic second-order differential operator in divergence form. Clearly, $\varrho_D = 1$ in (1.7). In addition, let $\varpi \neq \square$ and $A = \mathbb{I}_d$ so that $L(\nabla) = -\Delta$ is the negative Laplacian. In the case $B(x, \nabla) = \nu(x)^\top \nabla$ we have the Neumann spectral problem (1.4), (1.5), which describes, for example, the propagation of waves in a homogeneous acoustic medium polluted by particles of two types, $\square \setminus \varpi$ and $\square \setminus \varpi^\bullet$. A more general real, symmetric and positive definite matrix function A could describe an anisotropic and inhomogeneous medium, in particular a stratified one, if the periodicity occurs only in the directions $x_1, \dots, x_{d-1}$. The case of the Dirichlet boundary conditions is usually connected to quantum waveguides.

2°. *Elastic medium.* Let $d = n = 3$, $N = 6$ and

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 2^{-1/2}\partial_3 & 2^{-1/2}\partial_2 \\ 0 & \partial_2 & 0 & 2^{-1/2}\partial_3 & 0 & 2^{-1/2}\partial_1 \\ 0 & 0 & \partial_3 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_1 & 0 \end{pmatrix}, \quad \partial_j = \frac{\partial}{\partial x_j}. \tag{1.25}$$

This matrix is algebraically complete with $\varrho_D = 2$, see [37].

Using the Voigt-Mandel notation we regard the displacement vector u as the column $(u_1, u_2, u_3)^\top$, where u_j is the projection to x_j-axis. The strain column

$$\epsilon(u) = D(\nabla)u = \big(\varepsilon_{11}u, \varepsilon_{22}u, \varepsilon_{33}u, \sqrt{2}\varepsilon_{23}(u), \sqrt{2}\varepsilon_{31}(u), \sqrt{2}\varepsilon_{12}(u)\big)^\top \tag{1.26}$$

contains the Cartesian components of the strain tensor

$$\varepsilon_{jk}(u) = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big), \quad j, k = 1, 2, 3.$$

The elastic moduli of a deformable medium form the symmetric and positive definite 6×6-matrix $A(x)$, and by Hooke's law, it defines the stress column

$$\sigma(u; x) = A(x)\epsilon(u; x) = A(x)D(\nabla)u(x) \tag{1.27}$$

which has the same structure as (1.26). The normalization factors $2^{-1/2}$ and $\sqrt{2}$ in (1.25) and (1.26) make the natural norms of two representations, the tensor of rank 3 and the column of height 6, equal to each other. Finally, $M(x) = \varrho(x)\mathbb{I}_3$ in (1.6) is the material density.

The system (1.4) defined by (1.25)–(1.27) describes the time-harmonic oscillations of an elastic medium. The case of Neumann boundary conditions $B = N$, see (1.11), describes the case the surface $\partial\Pi$ is traction-free and the Dirichlet case $B = \mathbb{I}_d$ corresponds to the rigidly fixed surface. We also mention the linearized Signorini conditions which are obtained by fixing in (1.12) the orthogonal projection

$$P(x) = \nu(x)\nu(x)^\top.$$

These describe the situation that the solid Π is in an inseparable contact with the absolutely rigid profile $\partial\Pi$.

To treat two-dimensional problems of the elasticity theory one needs some obvious changes to the notation introduced above.

3°. In Section 4 we will describe other problems in mathematical physics which do not exactly satisfy the conditions although they can be treated by straightforward modifications of our approach.

2. Spectra of the problems.

2.1. **Purely periodic case.** The FBG-transform [11]

$$u(x) \mapsto U^\eta(x) = \frac{1}{(2\pi)^{d/2}} \sum_{\alpha \in \mathbb{Z}} e^{-i\alpha^\top \eta} u(x+\alpha) \tag{2.1}$$

is a discrete analogue of the Fourier transform, and it converts the problem (1.14) in the periodic set Π, (1.2), into the following problem in the periodicity cell:

$$a(U^\eta, V^\eta; \varpi) = \Lambda^\eta (MU^\eta, V^\eta)_\varpi. \tag{2.2}$$

Problem (2.2) depends on the Floquet parameter, the dual Gelfand variable

$$\eta = (\eta_1, \dots, \eta_d) \in [-\pi, \pi]^d, \tag{2.3}$$

and it is posed in the subspace

$$\begin{aligned} \mathcal{H}^\eta_{\rm per}(\varpi) = \big\{ & U^\eta \in H^1(\varpi)^n \,:\, PU^\eta = 0 \text{ on } \upsilon, \\ & U^\eta(x)\big|_{x_j = 1/2} = e^{i\eta_j} U^\eta(x)\big|_{x_j=-1/2},\ j = 1, \dots, d \big\}, \end{aligned} \tag{2.4}$$

the definition of which includes the Dirichlet part of the condition (1.6) restricted to $\upsilon \subset \partial\varpi$ and the so-called quasi-periodicity conditions on $\partial\varpi \cap \partial\square$, while Λ^η is just a new notation for the spectral parameter.

In the subspace (2.4) we introduce a new scalar product

$$\langle U, V \rangle_\varpi = a(U, V; \varpi) + (MU, V)_\varpi \tag{2.5}$$

and the positive, symmetric and continuous (thus self-adjoint) operator $\mathcal{A}^\eta$, which is determined by the relation

$$\langle \mathcal{A}^\eta U^\eta, V^\eta \rangle_\varpi = (M^\eta U^\eta, V^\eta)_\varpi \quad \forall\ U^\eta, V^\eta \in \mathcal{H}^\eta_{\rm per}(\varpi). \tag{2.6}$$

Our assumptions on A and M and the Korn inequality (1.8) imply that the sesquilinear form (2.5) is Hermitian, closed and positive in $\mathcal{H}^\eta_{\rm per}(\varpi)$.

Owing to (2.5) and (2.6), the problem (2.2) is equivalent to the abstract equation

$$\mathcal{A}^\eta U^\eta = \mu^\eta U^\eta \quad \text{in } \mathcal{H}^\eta_{\rm per}(\varpi)$$

with the new spectral parameter

$$\mu^\eta = (1 + \Lambda^\eta)^{-1}. \tag{2.7}$$

Due to the compactness of the embedding $H^1(\varpi) \subset L^2(\varpi)$, the operator $\mathcal{A}^\eta$ is compact and by [3, Thm. 10.1.5.,10.2.2.], [39, Thm. VI.16], its essential spectrum $\Sigma^\eta_{\rm ess}$ consists of the single point $\mu = 0$ and the discrete spectrum $\Sigma^\eta_{\rm di}$ of the positive monotonely decreasing sequence

$$1 \ge \mu^\eta_1 \ge \mu^\eta_2 \ge \ldots \ge \mu^\eta_m \ge \ldots \to +0,$$

where the multiplicities of the eigenvalues are taken into account.

The relation (2.7) determines the eigenvalue sequence

$$0 \leq \Lambda_1^\eta \leq \Lambda_2^\eta \leq \ldots \leq \Lambda_n^\eta \leq \ldots \to +\infty$$

of the problem (2.2). The corresponding eigenvectors $U_{(m)}^\eta \in \mathcal{H}_{\rm per}^\eta(\varpi)$, $m \in \mathbb{N}$, can be subject to the normalization and orthogonality conditions

$$\big(MU_{(m)}^\eta, U_{(k)}^\eta\big)_\varpi = \delta_{m,k}, \quad m, k \in \mathbb{N},$$

where $\delta_{m,k}$ is the Kronecker symbol. Each eigenpair $\{\Lambda_m^\eta, U_{(m)}^\eta\}$ generates the Floquet wave

$$w_{(m)}^{\eta m}(x) = e^{i\alpha^\top \eta} U_{(m)}^\eta (x - \alpha), \quad x \in \varpi(\alpha),\ \alpha \in \mathbb{Z}^d \tag{2.8}$$

which belongs to $H_{\rm loc}^1(\overline{\Pi})^n$ and satisfies the integral identity (1.14) with the parameter $\lambda = \Lambda_m^{\eta m}$ and test functions $v \in C_c^\infty(\overline{\Pi})^n \cap \mathcal{H}(\Pi)$.

The functions $[-\pi, \pi] \ni \eta \mapsto \Lambda_m^\eta$ are continuous and 2π-periodic in the variables (2.3) for all $m \in \mathbb{N}$. This fact, other main properties of the FBG-transform and information on purely periodic elliptic problems can be found in the monographs [44, 18, 39] and others. In particular it is known that the spectrum σ of the original problem has the band-gap structure

$$\sigma = \bigcup_{m \in \mathbb{N}} \beta_m \tag{2.9}$$

composed of the spectral bands β_m, which are compact intervals

$$\beta_m = \{\Lambda_m^\eta \,:\, \eta \in [-\pi, \pi]^d\}, \quad m \in \mathbb{N}. \tag{2.10}$$

If Λ_m^η does not depend on $\eta \in [-\pi, \pi]^d$, i.e. $\Lambda_m^\eta = \Lambda_m^0$ for all η, then Λ_m^0 is an eigenvalue of the problem (1.14) with infinite multiplicity and it belongs to the point spectrum $\sigma_{\rm po}$. For particular scalar problems with certain geometric restrictions it has been proved that $\sigma_{\rm po} = \varnothing$, see [42, 43, 41, 17] and others. There exist examples of concrete problems with non-empty point spectrum. The discrete spectrum $\sigma_{\rm di}$ of purely periodic elliptic problems is always empty.

The band-gap structure (2.9) makes it possible that a spectral gap $\gamma_m \neq \varnothing$ is opened between the bands β_m and β_{m+1}, the gap being an open interval, which is free of the essential spectrum but has both endpoints in it. In the case that the lower bound

$$\lambda_\dagger := \underline{\sigma} = \min\{\Lambda_0^\eta \,:\, \eta \in [-\pi, \pi]^d\} \tag{2.11}$$

is positive, we will also consider the interval $\gamma_0 = (0, \lambda_\dagger)$ below the essential spectrum. Examples of spectral gaps exist in the case of lattices of thin acoustic and quantum waveguides (see [38, 7, 19, 12, 34, 2, 27, 28] and many others), in double porocity problems [13, 14, 46], in the Dirichlet and Neumann problems for the Laplacian in the periodically perforated plane [32, 33, 8] and other problems in mathematical physics.

2.2. **The case of a local perturbation.** We now consider problem (1.18), which concerns the periodic medium Π° perturbed in one cell, see (1.16), and which is posed in the space

$$\mathcal{H}(\Pi^\circ) = \{u \in H^1(\Pi^\circ)^n \,:\, Pu = 0 \text{ on } \partial\Pi \setminus \upsilon,\ P^\bullet u = 0 \text{ on } \upsilon^\bullet \}$$

with the scalar product

$$\langle u^\circ, v^\circ \rangle_\circ = a^\circ(u^\circ, v^\circ; \Pi^\circ) + (M^\circ u^\circ, v^\circ)_{\Pi^\circ} \tag{2.12}$$

(cf. (2.5) and (2.6)), and the positive, continuous self-adjoint operator $\mathcal{A}^\circ$, which is determined by the relation

$$\langle \mathcal{A}^\circ u^\circ, v^\circ \rangle_\varpi = (M^\circ u^\circ, v^\circ)_\varpi \quad \forall\, u^\circ, v^\circ \in \mathcal{H}(\Pi^\circ). \tag{2.13}$$

Problem (1.18) becomes equivalent to the abstract equation

$$\mathcal{A}^\circ u^\circ = \mu^\circ u^\circ \quad \text{in } \mathcal{H}(\Pi^\circ)$$

with the spectral parameter

$$\mu^\circ = (1 + \lambda^\circ)^{-1}, \tag{2.14}$$

cf. (2.7). Since the perturbation is localized, the essential spectrum $\sigma^\circ_{\rm ess}$ of the problem (1.18) coincides with that of the unperturbed problem, hence,

$$\sigma^\circ_{\rm ess} = \sigma \quad \text{and} \quad \Sigma^\circ_{\rm ess} = \{0\} \cup \big\{\mu^\circ \,:\, (\mu^\circ)^{-1} - 1 \in \sigma^\circ_{\rm ess}\big\}, \tag{2.15}$$

where $\Sigma^\circ_{\rm ess}$ is the essential spectrum of the operator $\mathcal{A}^\circ$. However, in contrast to the purely periodic problem (1.14), the discrete components

$$\sigma^\circ_{\rm di} \quad \text{and} \quad \Sigma^\circ_{\rm di} = \big\{\mu^\circ \,:\, (\mu^\circ)^{-1} - 1 \in \sigma^\circ_{\rm di}\big\}.$$

of the spectra σ° and Σ° may be nonempty.

Examples of eigenvalues inside spectral gaps γ_m, $m \ge 1$, and on the interval γ_0 below the essential spectrum (2.15) can be found, e.g., in [9, 1, 4, 6, 29].

Let us demonstrate by a standard example of a perturbation of a periodic medium, how one can find an eigenvalue in any spectral gap. We consider a perforated medium (1.2), that is, we set $\varpi = \square \setminus \overline{\omega}$, where $\omega \neq \varnothing$ is a domain with a smooth boundary and $\overline{\omega} \subset \square$. We also pick up another domain $\omega^\bullet$ with a smooth boundary such that $\overline{\omega^\bullet} \subset \omega$, and connect the original cell ϖ with $\omega^\bullet$ by a cylindrical ligament ς^ε with a circular cross-section of radius ε. We set

$$\begin{aligned} &\varpi^\bullet = \varpi \cup \omega^\bullet \cup \varsigma^\varepsilon, \quad \mathcal{A}^\bullet = \mathcal{A} \text{ in } \varpi^\bullet \setminus \omega^\bullet, \quad \mathcal{A}^\bullet = \mathbb{I}_N \text{ in } \omega^\bullet, \\ &M^\bullet = M,\ P^\bullet = P \text{ in } \varpi^\bullet \setminus \omega^\bullet, \quad M^\bullet = \rho \mathbb{I}_n,\ P^\bullet = \mathbb{I}_n \text{ in } \omega^\bullet, \end{aligned} \tag{2.16}$$

Lemma 2.1. *1) Let $P = \mathbb{I}_n$ so that* (1.5) *coincides with the Dirichlet condition. Then, the cut-off value* (2.11) *is positive.*

2) Assume that the spectral gap γ_m, $m \in \{0\} \cup \mathbb{N}$, is non-empty. Then, one can find the parameters ε and ρ in (2.16) *such that $\gamma_m \neq \varnothing$ contains at least one eigenvalue of the operator $\mathcal{A}^\circ$.*

Proof. 1) Owing to the Dirichlet condition on υ (see the explanation below (1.12)), the Korn inequality (1.8) becomes

$$\|u; H^1(\varpi)\|^2 \le c_{\varpi,A}\, a(u,u;\varpi) \quad \forall\, u \in H^1(\varpi)^n,\ u = 0 \text{ on } \upsilon. \tag{2.17}$$

Estimates (2.17) and (1.15) yield

$$a(u, u; \varpi) \geq c_{\varpi,A}^{-1} \|u; L^2(\varpi)\|^2 \geq (c_{\varpi,A} C_M)^{-1} (Mu, u)_\varpi$$

and thus also $\lambda_\dagger > (c_{\varpi,A} C_M)^{-1} > 0$.

2) Let $\tau_1^\bullet > 0$ be the first eigenvalue of the Dirichlet problem

$$\overline{D(-\nabla)}^\top D(\nabla) w^\bullet(x) = \tau^\bullet w^\bullet(x),\ x \in \omega^\bullet, \quad w^\bullet(x) = 0,\ x \in \partial\omega^\bullet, \tag{2.18}$$

cf. the notation in (2.16). The corresponding eigenvector $w_{(1)}^\bullet$ belongs to $C^\infty(\overline{\omega^\bullet})^n$ due to our assumption on the smoothness of the boundary $\partial\omega^\bullet$, and therefore

$$|w^\bullet(x)| \leq c \operatorname{dist}(x, \partial\omega^\bullet), \quad |\nabla w^\bullet(x)| \leq c,\ x \in \overline{\omega^\bullet}. \tag{2.19}$$

Let Q be the intersection point of the axis of the circular cylinder ζ^ε and the surface $\partial\omega\bullet$. We define the smooth cut-off functions X and X_ε,

$$\begin{aligned} &X(r) = 1 \text{ for } r \geq 3 \text{ and } X(r) = 0 \text{ for } r \leq 2, \\ &X_\varepsilon = X(\varepsilon^{-1}|x - Q|),\ x \in \omega^\bullet \text{ and } X_\varepsilon = 0,\ x \in \Pi^\circ \setminus \omega^\bullet. \end{aligned} \tag{2.20}$$

Then, we set $u^\varepsilon = X_\varepsilon w^\bullet \in \mathcal{H}(\Pi^\circ)$ and observe that in view of (2.19) and (2.20) we have

$$\begin{aligned} &\Big|(Mu^\varepsilon, u^\varepsilon)_{\Pi^\circ} - \|w^\bullet; L^2(\omega^\bullet)\|^2\Big| \leq c\varepsilon^{2+d}, \\ &\Big|a^\circ(u^\varepsilon, u^\varepsilon; \Pi^\circ) - \|D(\nabla) w^\bullet; L^2(\omega^\bullet)\|^2\Big| \leq c\varepsilon^d. \end{aligned} \tag{2.21}$$

Our lemma will follow from a well-known perturbation result, namely the lemma on "near eigenvalues" [45], which is based on the spectral decomposition of the resolvent (see e.g. [3, Ch. 6]). It can be written briefly as the implication

$$\begin{aligned} &\|\mathcal{U}^\circ; \mathcal{H}(\Pi^\circ)\| = 1,\ \mathcal{M}^\circ \in \mathbb{R}_+, \|\mathcal{A}^\circ\mathcal{U}^\circ - \mathcal{M}^\circ\mathcal{U}^\circ; \mathcal{H}(\Pi^\circ)\| =: \delta^\circ \in (0, \mathcal{M}^\circ) \\ &\Rightarrow \exists\ \mu^\circ \in \sigma^\circ \text{ such that } |\mu^\circ - \mathcal{M}^\circ| \leq \delta^\circ. \end{aligned} \tag{2.22}$$

We recall the factor ρ in (2.16) as well as the relation (2.14) and take $\mathcal{U}^\circ = \|u^\varepsilon; \mathcal{H}(\Pi^\circ)\|^{-1} u^\varepsilon$ and $\mathcal{M}^\circ = (1 + \rho^{-1}\tau_1^\bullet)^{-1}$. Using the definition of the norm of a Hilbert space, we obtain

$$\begin{aligned} \delta_0 &:= \|\mathcal{A}^\circ\mathcal{U}^\circ - \mathcal{M}^\circ\mathcal{U}^\circ; \mathcal{H}(\Pi^\circ)\| = \inf \big|\langle \mathcal{A}^\circ\mathcal{U}^\circ - \mathcal{M}^\circ\mathcal{U}^\circ, v^\circ\rangle\big| \\ &= (1 + \rho^{-1}\tau_1^\bullet)^{-1} \|u^\varepsilon; \mathcal{H}(\Pi^\circ)\|^{-1} \\ &\times \inf \Big|(1 + \rho^{-1}\tau_1^\bullet)(M^\circ u^\varepsilon, v^\circ)_{\Pi^\circ} - a^\circ(u^\varepsilon, v^\circ; \Pi^\circ) - (M^\circ u^\varepsilon, v^\circ)_{\Pi^\circ}\Big| \\ &= (1 + \rho^{-1}\tau_1^\bullet)^{-1} \|u^\varepsilon; \mathcal{H}(\Pi^\circ)\|^{-1} \inf \Big|a^\circ(u^\varepsilon, v^\circ; \Pi^\circ) - (\tau_1^\circ)^{-1}(M^\circ u^\varepsilon, v^\circ)_{\Pi^\circ}\Big|. \end{aligned}$$

Here, the infimum is taken over all functions $v^\circ \in \mathcal{H}(\Pi^\circ)$ with norm one, and we have also used formulas (2.12) and (2.13). Taking into account (2.16), (2.21) and (2.18) yields

$$\begin{aligned} &\Big|(a^\circ(u^\varepsilon, v^\circ; \Pi^\circ) - \tau_1^\circ(\rho^{-1} M^\circ u^\varepsilon, v^\circ)_{\Pi^\circ}) \\ &-\big((D(\nabla) w_{(1)}^\bullet, D(\nabla)(X_\varepsilon v^\circ)\big)_{\omega^\bullet} - \tau_1^\circ (w_{(1)}, X_\varepsilon v^\circ)_{\omega^\bullet}\Big| \\ &\leq c\varepsilon^{d/2} \|v^\circ; \mathcal{H}(\Pi^\circ)\| \end{aligned}$$

and thus $\delta_0 \leq c_0\varepsilon^{d/2}$.

We now choose $\rho > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} &\big[(1+\rho^{-1}\tau_1^\circ)^{-1} - c_0\varepsilon^{d/2}, (1+\rho^{-1}\tau_1^\circ)^{-1} + c_0\varepsilon^{d/2}\big] \subset \Gamma_m \\ &= \{\mu \,:\, \mu^{-1} - 1 \in \gamma_m\} \neq \varnothing \end{aligned}$$

and use assertion (2.22) to complete the proof. □

Remark 2.2. 1) In the above application of the implication (2.22) it was much more convenient to use the operator $\mathcal{A}^\circ$, (2.13), than the operator directly connected to the variational problem (1.18). The same will also happen in Section 3.

2) In the case $\lambda_\dagger > 0$, cf. Lemma 2.1.1), there exists a much easier way to verify that $\gamma_0 \cap \sigma_{\mathrm{di}}^\circ \neq \varnothing$ with the help of the following observation. Namely, the norm $\mathbf{n}^\circ < 1$ of the operator $\mathcal{A}^\circ$ is an eigenvalue belonging to the discrete spectrum $\Sigma_{\mathrm{di}}^\circ$, if the upper bound $(1+\lambda_\dagger)^{-1} < 1$ of the essential spectrum $\Sigma_{\mathrm{ess}}^\circ$ is strictly less than $\mathbf{n}^\circ$. Then, by (2.14), $(\mathbf{n}^\circ)^{-1} - 1 \in \gamma_0 \cap \sigma_{\mathrm{di}}^\circ$. In the above example, we take the zero extension $u^\bullet$ of $w_{(1)}^\bullet$ from $\omega^\bullet$ onto Π°: this falls into the space $\mathcal{H}(\Pi^\circ)$ and therefore

$$\begin{aligned} \mathbf{n}^\circ &= \sup_{u^\circ \in \mathcal{H}(\Pi^\circ)} \frac{\langle \mathcal{A}^\circ u^\circ, u^\circ\rangle_\circ}{\langle u^\circ, u^\circ\rangle_\circ} \geq \frac{\langle \mathcal{A}^\circ u^\bullet, u^\bullet\rangle_\circ}{\langle u^\bullet, u^\bullet\rangle_\circ} \\ &= \frac{\rho\|w_{(1)}^\bullet; L^2(\omega^\bullet)\|^2}{\|D(\nabla)w_{(1)}^\bullet; L^2(\omega^\bullet)\|^2 + \rho\|w_{(1)}^\bullet; L^2(\omega^\bullet)\|^2} = \frac{\rho}{\tau_1^\bullet + \rho}. \end{aligned}$$

It again suffices to choose $\rho > 0$ properly.

2.3. **The case of sparsely placed inclusions.** In the same way as above we introduce the scalar product

$$\langle u^\sharp, v^\sharp\rangle_\sharp = a^\sharp(u^\sharp, v^\sharp; \Pi^\sharp) + (M^\sharp u^\sharp, v^\sharp)_{\Pi^\sharp}$$

and the positive, continuous, self-adjoint operator $\mathcal{A}^\sharp$, which is determined by the formula

$$\langle \mathcal{A}^\sharp u^\sharp, v^\sharp\rangle_\varpi = (M^\sharp u^\sharp, v^\sharp)_{\Pi^\sharp} \quad \forall\, u^\sharp, v^\sharp \in \mathcal{H}(\Pi^\sharp).$$

in the space

$$\mathcal{H}(\Pi^\sharp) = \{u^\sharp \in H^1(\Pi^\sharp)^n \,:\, P^\sharp u^\sharp = 0 \text{ on } \partial\Pi^\sharp\ \}.$$

The relation of the essential spectrum $\Sigma_{\mathrm{ess}}^\sharp \subset [0,1]$ of the operator $\mathcal{A}^\sharp$ and the essential spectrum $\sigma_{\mathrm{ess}}^\sharp$ of the problem (1.22) is the same as before,

$$\sigma_{\mathrm{ess}}^\sharp = \{\lambda \,:\, (1+\lambda)^{-1} \in \Sigma_{\mathrm{ess}}^\sharp\}. \tag{2.23}$$

In the following, main result of this paper we characterize the essential spectrum of the principal problem formulated in Section 1.3.

Theorem 2.3. *The essential spectrum* (2.23) *of the boundary value problem* (1.20)–(1.21)*, or* (1.22)*, in the medium* (1.19) *with sparsely placed inclusions* (1.3) *equals*

$$\sigma_{\mathrm{ess}}^\sharp = \sigma \cup \sigma_{\mathrm{di}}^\circ$$

where σ is the spectrum of the problem (1.14) *in the purely periodic medium* (1.2) *and $\sigma_{\mathrm{di}}^\circ$ is the discrete spectrum of the problem* (1.18) *in the medium* (1.16) *with the single inclusion $\varpi^\bullet$.*

Recall that the spectra of the problems (1.14) and (1.18) were described in Sections 2.1 and 2.2, respectively.

The proof will be given in the next section.

2.4. **Operators of the inhomogeneous problems.** Let us fix the parameter $\lambda \in \mathbb{C}$. In addition to the operators $\mathcal{A}$ and $\mathcal{A}^\circ$, we will also need the operators

$$\mathcal{B}(\lambda) : \mathcal{H}(\Pi) \to \mathcal{H}(\Pi)^* \quad \text{and} \quad \mathcal{B}^\circ(\lambda) : \mathcal{H}(\Pi^\circ) \to \mathcal{H}(\Pi^\circ)^*, \tag{2.24}$$

respectively, related to the spectral problems (1.4)–(1.5), (1.14), on the intact domain Π and to (1.18) on the perturbed domain Π°. The operator $\mathcal{B}(\lambda)$ is defined by mapping $u \in \mathcal{H}(\Pi)$ to the functional

$$v \mapsto a(u, v; \Pi) - \lambda(Mu, v)_\Pi, \quad v \in \mathcal{H}(\Pi), \tag{2.25}$$

and the definition of $\mathcal{B}^\circ(\lambda)$ is analogous. In other words, $\mathcal{B}(\lambda)$ is the problem operator of the inhomogeneous problem

$$a(u, v; \Pi) - \lambda(Mu, v)_\Pi = f(v) \quad \forall\, v \in \mathcal{H}(\Pi), \tag{2.26}$$

where $f \in H(\Pi)^*$ is given; the case $\mathcal{B}^\circ(\lambda)$ is similar.

Lemma 2.4. *If $\lambda \in \mathbb{R}$, $\lambda \notin \sigma = \sigma_{\mathrm{ess}}$, then the operator $\mathcal{B}(\lambda) : \mathcal{H}(\Pi) \to \mathcal{H}(\Pi)^*$ is an isomorphism.*

Proof. The FBG-transform is an isomorphism from the Sobolev space $H^1(\Pi)$ onto the space $L^2([-\pi, \pi]^d; \mathcal{H}^\eta_{\mathrm{per}}(\varpi))$ of abstract functions in η with the norm

$$\Big(\int\limits_{[-\pi,\pi]^d} \|U^\eta; \mathcal{H}^\eta_{\mathrm{per}}(\varpi)\|^2 d\eta \Big)^{1/2}.$$

Applying the FBG-transform, equation (2.26) turns into the problem,

$$a(U^\eta, V^\eta; \varpi) - \lambda(MU^\eta, V^\eta)_\varpi = F^\eta(V^\eta) \quad \forall\, V^\eta \in \mathcal{H}^\eta(\varpi), \tag{2.27}$$

where the notation is as in (2.1), and F^η is defined as the compose of f and theFBG-transform. By the assumption, the distance of λ from the union of the eigenvalues Λ^η (see (2.9), (2.10)) is positive, hence, (2.27) has a unique solution U^η for every η, and we even get an upper bound for the norm $\|U^\eta; H^1(\varpi)\|$. Taking the inverse FBG-transform yields a solution u of (2.26).

The uniqueness of the solution by an indirect argument: having two different solutions of (2.26) would lead to having two different solutions of (2.27) for some η, which is a contradiction. Finally, as $\mathcal{B}(\lambda)$ is a bounded operator, the boundedness of the inverse follows from the open mapping theorem. $\square$

3. Identification of the essential spectrum.

3.1. **On exponentially decaying solutions of the inhomogeneous problems.** Let us proceed to consider the inhomogeneous problem

$$a^\circ(u^\circ, v^\circ; \Pi^\circ) - \lambda(M^\circ u^\circ, v^\circ)_{\Pi^\circ} = f^\circ(v^\circ) \quad \forall\, v^\circ \in \mathcal{H}(\Pi^\circ), \tag{3.1}$$

where $f^\circ \in \mathcal{H}(\Pi^\circ)^*$ is an (anti)linear continuous functional on $\mathcal{H}^\circ(\Pi^\circ)$. In this section we treat the parameter values

$$\lambda \in \mathbb{R}, \ \lambda \notin \sigma^\circ_{\mathrm{ess}} = \sigma \tag{3.2}$$

and the operator $\mathcal{B}^\circ(\lambda) : \mathcal{H}(\Pi^\circ) \to \mathcal{H}(\Pi^\circ)^*$ of (2.24).

Proposition 3.1. *If* (3.2) *holds, then the operator* $\mathcal{B}^\circ(\lambda)$*,* (2.24)*, is Fredholm.*

This result follows immediately from the following one.

Lemma 3.2. *If* (3.2) *holds, the operator* $\mathcal{B}^\circ(\lambda)$ *has a parametrix, i.e. a mapping* $\mathcal{R}^\circ(\lambda) : \mathcal{H}(\Pi^\circ)^* \to \mathcal{H}(\Pi^\circ)$ *such that the operator*

$$\mathcal{B}^\circ(\lambda)\mathcal{R}^\circ(\lambda) - \mathrm{id} : \mathcal{H}(\Pi^\circ)^* \to \mathcal{H}(\Pi^\circ)^* \tag{3.3}$$

is compact.

The compactness of (3.3) actually means that $\mathcal{R}^\circ(\lambda)$ is a right parametrix for $\mathcal{B}^\circ(\lambda)$, but this suffices, since $\mathcal{B}^\circ(\lambda)$ is self-adjoint with respect to the duality so that the adjoint of $\mathcal{R}^\circ(\lambda)$ is a left parametrix.

Proof. Let f° be as in (3.1). We recall that by Lemma 2.4 and assumption (3.2), the operator $\mathcal{B}(\lambda)$ is an isomorphism. The parametrix will be constructed as a perturbation of the inverse $\mathcal{B}(\lambda)^{-1}$ of $\mathcal{B}(\lambda)$. Putting $\mathcal{B}(\lambda)^{-1}$ into (3.3) instead of $\mathcal{R}^\circ(\lambda)$ leaves a discrepancy in a neighborhood of the modified cell $\varpi^\bullet$, which is compensated by solving a variational problem (see (3.8), below) in a bounded domain and extending its solution to Π° by using suitable cut-off functions.

1°. In the first part of the proof we define three subsidiary functionals. We define the smooth cut-off functions

$$\chi(t) = 1 \text{ for } t > 1/4 \text{ and } \chi(t) = 0 \text{ for } t < -1/4,$$
$$X_J(x) = \prod_{j=1}^{d} \chi(J + x_j)\chi(J - x_j), \quad J \in \mathbb{N}. \tag{3.4}$$

Notice that $X_J \in C_c^\infty(\square_J)$ and $X_J = 1$ on $\square_{J-1}$, where

$$\square_J = \{x : |x_j| < J + 1/2,\ j = 1, \dots, d\}. \tag{3.5}$$

Let us define the functional $f^1 \in \mathcal{H}(\Pi)^*$ and the vector function $u^1 \in \mathcal{H}(\Pi) \cap \mathcal{H}(\Pi^\circ)$ by

$$f^1(v) = f^\circ\big((1 - X_2)v\big), \quad u^1 = (1 - X_2)\mathcal{B}(\lambda)^{-1} f^1. \tag{3.6}$$

We next set

$$\begin{aligned} f^2(v^\circ) \;=\; & f^\circ\big((1 - (1 - X_2)^2)v^\circ\big) - \big(A^\circ u^1 D(\nabla) X_2, D(\nabla) v^\circ\big)_{\Pi^\circ} \\ & + \big(A^\circ D(\nabla) u^1, v^\circ D(\nabla) X_2\big)_{\Pi^\circ}. \end{aligned} \tag{3.7}$$

Since $1 - (1 - X_2)^2 = X_2(2 - X_2)$, this functional has a compact support in $\overline{\Pi_2^\circ}$, where $\Pi_N^\circ = \Pi^\circ \cap \square_N$ for $N \in \mathbb{N}$. Then, we consider the problem

$$\begin{aligned} & a_T^0(u^2, v^\circ; \Pi_3^\circ) = f^2(v^\circ), \quad \text{where} \\ & a_T^0(u^2, v^\circ; \Pi_3^\circ) := a^\circ(u^2, v^\circ; \Pi_3^\circ) - \lambda(M^\circ u^2, v^\circ)_{\Pi_3^\circ} + T(u^2, v^\circ)_{\Pi_3^\circ}, \end{aligned} \tag{3.8}$$

which is posed in the space

$$\begin{aligned} & \mathcal{H}_\circ(\Pi_J^\circ) \\ & = \big\{v \in H^1(\Pi_J^\circ) : P^\circ v = 0 \text{ on } \partial\Pi^\circ \cap \square_J,\ v = 0 \text{ on } \partial\square_J \cap \Pi^\circ\big\} \end{aligned} \tag{3.9}$$

with $J = 3$. Problem (3.9) is uniquely solvable for large $T > 0$. This follows from the Lax-Milgram lemma, since summing the Korn inequalities (1.8) in $\varpi(\alpha) \subset \Pi_3$, $\alpha \neq 0$, and the inequality

$$\|v^\circ; H^1(\varpi^\bullet)\| \leq c_{\varpi^\bullet}\big(a^\circ(v^\circ, v^\circ; \varpi^\bullet) + \|v^\circ; L^2(\varpi^\bullet)\|^2\big),$$

we see that

$$\begin{aligned} a_T^0(v^\circ, v^\circ; \Pi_3^\circ) \geq \ &\min\{c_\varpi^{-1}, c_{\varpi^\bullet}^{-1}\}\|v^\circ; H^1(\Pi_3^\circ)\|^2 \\ &+(T - 1 - \lambda C_M)\|v^\circ; L^2(\Pi_3^\circ)\|^2. \end{aligned}$$

We still denote

$$\begin{aligned} f^3(v^\circ) = \ &T(X_3 u^2, v^\circ)_{\Pi^\circ} + \big(A^\circ u^2 D(\nabla) X_3, D(\nabla) v^\circ\big)_{\Pi^\circ} \\ &-\big(A^\circ D(\nabla) u^2, v^\circ D(\nabla) X_3\big)_{\Pi^\circ} \end{aligned} \tag{3.10}$$

By standard estimates,

$$\|u^1; \mathcal{H}(\Pi^\circ)\| + \|X_3 u^2; \mathcal{H}(\Pi^\circ)\| \leq c\|f^\circ; \mathcal{H}(\Pi^\circ)^*\|$$

hence, we observe that, first,

$$\|f^3; \mathcal{H}(\Pi^\circ)^*\| \leq c\|f^\circ; \mathcal{H}(\Pi^\circ)^*\|$$

and, second,

(3.11) the mapping $f^\circ \mapsto f^3$ is compact in the space $\mathcal{H}(\Pi^\circ)^*$,

because each of the scalar products on the right-hand side of (3.10) has the compactly supported factor X_3 and contains derivatives of u^2 or v° at most in one position.

2°. We now show that the parametrix can be defined by $\mathcal{R}^0(\lambda)f^\circ = u^1 + X_3 u^2$. Notice that the calculation (3.12)–(3.14), with straightforward changes, will be used several times in the sequel.

We calculate for all $v^\circ \in \mathcal{H}(\Pi^\circ)$ using (2.25), (3.6) and taking into account the support of the cut-off function,

$$\begin{aligned} \mathcal{B}^\circ(u^1)(v^\circ) = \ &\Big(AD(\nabla)\big((1 - X_2)\mathcal{R}(\lambda)^{-1} f^1\big), D(\nabla) v^\circ\Big)_\Pi \\ &-\lambda\big(M(1 - X_2)\mathcal{R}(\lambda)^{-1} f^1, v^\circ\big)_\Pi \end{aligned} \tag{3.12}$$

We commute here the cut-off function $1 - X_2$ to the right factor so that (3.12) equals

$$\begin{aligned} &\Big(AD(\nabla)\mathcal{R}(\lambda)^{-1} f^1, D(\nabla)\big((1 - X_2)v^\circ\big)\Big)_\Pi \\ &-\lambda\big(M(\mathcal{R}(\lambda)^{-1} f^1, (1 - X_2)v^\circ\big)_\Pi \\ &+\big(A^\circ u^2 D(\nabla) X_2, D(\nabla) v^\circ\big)_{\Pi^\circ} - \big(A^\circ D(\nabla) u^2, v^\circ D(\nabla) X_2\big)_{\Pi^\circ}. \end{aligned} \tag{3.13}$$

Here we use (2.25) for the first two terms and (3.7) for the last ones: (3.12) equals

$$\begin{aligned} &\mathcal{B}(\lambda)\mathcal{R}(\lambda) f^1\big((1 - X_2)v^\circ\big) + f^\circ\big((1 - (1 - X_2)^2)v^\circ\big) \\ &= f^1((1 - X_2)v^\circ) - f^2(v^\circ) + f^\circ\big((1 - (1 - X_2)^2)v^\circ\big) = f^\circ(v^\circ) - f^2(v^\circ). \end{aligned}$$

Hence,

$$\mathcal{B}^\circ u^1 = f^\circ - f^2. \tag{3.14}$$

Commuting the cut-off function in the same way as in (3.12)-(3.13) and taking into account (3.8), (3.10) yield

$$\begin{aligned}
&\mathcal{B}^\circ(X_3u^2)(v^\circ)\\
&= \big(A^\circ D(\nabla)(X_3u^2), D(\nabla)v^\circ\big)_{\Pi_3^\circ} - \lambda\big(M^\circ(X_3u^2), v^\circ\big)_{\Pi_3^\circ}\\
&= \big(A^\circ D(\nabla)u^2, D(\nabla)(X_3v^\circ)\big)_{\Pi_3^\circ} - \lambda\big(M^\circ u^2, (X_3v^\circ)\big)_{\Pi_3^\circ}\\
&\quad +\big(A^\circ u^2 D(\nabla)X_3, D(\nabla)v^\circ\big)_{\Pi_3^\circ} - \big(A^\circ D(\nabla)u^2, v^\circ D(\nabla)X_3\big)_{\Pi_3^\circ}\\
&= f^2(v^\circ) + T(u^2, v^\circ)_{\Pi_3^\circ}\\
&\quad +\big(A^\circ u^2 D(\nabla)X_3, D(\nabla)v^\circ\big)_{\Pi_3^\circ} - \big(A^\circ D(\nabla)u^2, v^\circ D(\nabla)X_3\big)_{\Pi_3^\circ}\\
&= f^2(v^\circ) + f^3(v^\circ) \quad \Rightarrow \quad \mathcal{B}^\circ(X_3u^2) = f^2 + f^3.
\end{aligned} \tag{3.15}$$

In view of (3.14), (3.15), (3.11), setting $\mathcal{R}^\circ(\lambda)f^\circ = u^1 + X_3u^2$ yields the desired parametrix. □

Our immediate goal is now to prove that a solution $u^\circ \in \mathcal{H}(\Pi^\circ)$ of the problem (3.1) inherits the exponential decay at infinity from the right-hand side f°.

We fix the spectral parameter λ satisfying (3.2). The subspace $\ker \mathcal{B}^\circ(\lambda)$, which consists of the solutions of the homogeneous problem (1.18), may be non-trivial, since λ may still be an eigenvalue. However, the dimension $K = \dim \ker\mathcal{B}^\circ(\lambda)$ is finite due to Proposition 3.1. Let $u^\circ_{(1)}, \dots, u_{(K)} \in \mathcal{H}(\Pi^\circ)$ be a basis of $\ker \mathcal{B}^\circ(\lambda)$. We can choose the number $J \in \mathbb{N}$ such that the restrictions $u^\circ_{(1)}\big|_{\Pi_J^\circ}, \dots, u^\circ_{(K)}\big|_{\Pi_J^\circ}$ are linearly independent in $L^2(\Pi_J^\circ)^n$. Moreover, we find vector functions $\psi_{(1)}, \dots, \psi_{(K)} \in \mathcal{H}(\Pi_J^\circ)$ (see (3.9)) such that

$$\big(u^\circ_{(j)}, \psi_{(k)}\big)_{\Pi_J^\circ} = \delta_{j,k}, \quad j, k = 1, \dots, K. \tag{3.16}$$

Proposition 3.3. *For every λ as in (3.2), the problem*

$$\begin{aligned}
a^\circ(u^\flat, v^\flat; \Pi^\circ) \,-\, \lambda(M^\circ u^\flat, v^\flat)_{\Pi^\circ} + \sum_{k=1}^{K} (\psi_{(k)}, v^\flat)_{\Pi_J^\circ} (u^\flat, \psi_{(k)})_{\Pi_J^\circ}&\\
= f^\flat(v^\flat) \quad \forall\, v^\flat \in \mathcal{H}(\Pi^\circ)&
\end{aligned} \tag{3.17}$$

has a unique solution $u^\flat \in \mathcal{H}(\Pi^\circ)$ for every $f^\flat \in \mathcal{H}(\Pi^\circ)^*$, and there holds the estimate

$$\|u^\flat; \mathcal{H}(\Pi^\circ)\| \le c_\flat \|f^\flat; \mathcal{H}(\Pi^\circ)^*\|$$

Proof. The functional

$$v \mapsto f^\circ(v) = f^\flat(v) - \sum_{k=1}^{K} f^b(u^\circ_{(k)})(\psi_{(k)}, v)_{\Pi_J^\circ},$$

clearly has the property

$$f^\circ(v) = 0 \quad \forall\, v \in \ker \mathcal{B}^\circ(\lambda).$$

Since the operator (2.24) is self-adjoint, problem (3.1) with the right-hand side f° thus has a solution $u^\circ \in \mathcal{H}(\Pi^\circ)$ which is defined up to an addendum belonging to $\ker \mathcal{B}^\circ(\lambda)$, i.e.,

$$u^\flat = u^\circ + c_1 u^\circ_{(1)} + \ldots + c_K u^\circ_{(K)}.$$

The conditions $(u^\circ, \psi_{(k)})_{\Pi^\circ_J} = 0$, $k = 1, \ldots, K$ make the solution u° unique. It remains to set

$$c_K = f^\flat(u_{(k)}), \quad k = 1, \ldots, K.$$

□

Let us now introduce the weighted space $\mathcal{W}_\kappa(\Pi^\circ)$ of vector functions $u^\circ \in H^2_{\rm loc}(\overline{\Pi^\circ})^n$ which fulfill the boundary condition $P^\circ u^\circ = 0$ on $\partial\Pi^\circ$ and have finite norm

$$\|u^\circ; \mathcal{W}_\kappa(\Pi^\circ)\| = \|\mathcal{E}_\kappa u^\circ; H^1(\Pi^\circ)\|,$$

where the weight function $\mathcal{E}_\kappa$ is defined for all $\kappa \geq 0$ by

$$\mathcal{E}_\kappa(x) = \prod_{j=1}^{d} \max\{1, e^{\kappa(|x_j| - J - 1/2)}\}, \tag{3.18}$$

and by $\mathcal{E}_\kappa(x) = \mathcal{E}_{|\kappa|}(x)^{-1}$ for $\kappa < 0$. The number J is fixed such that $\mathcal{E}_\kappa = 1$ on $\operatorname{supp} \psi_{(k)} \subset \overline{\Pi^\circ_J}$.

For $\kappa > 0$, any solution $u^\kappa := u^\flat \in \mathcal{W}_\kappa(\Pi^\circ) \subset \mathcal{H}(\Pi^\circ)$ of the problem (3.17) must by definition satisfy the integral identity

$$\begin{aligned} a^\circ(u^\kappa, v^\kappa; \Pi^\circ) \, - \, \lambda(M^\circ u^\kappa, v^\kappa)_{\Pi^\circ} + \sum_{k=1}^{K} (\psi_{(k)}, v^\kappa)_{\Pi^\circ}(u^\kappa, \psi_{(k)})_{\Pi^\circ} \\ = f^\kappa(v^\kappa) \quad \forall \, v^\kappa \in \mathcal{W}_{-\kappa}(\Pi^\circ), \end{aligned} \tag{3.19}$$

where

$$f^\kappa \in \mathcal{W}_{-\kappa}(\Pi^\circ)^* \tag{3.20}$$

is a continuous (anti)linear functional $\mathcal{W}_{-\kappa}(\Pi^\circ) \to \mathbb{C}$, which decays exponentially at infinity, since it acts on the space of exponentially growing vector functions.

Theorem 3.4. *For every λ as in* (3.2)*, there exists $\kappa^0 > 0$ such that for $\kappa \in [0, \kappa^0)$, problem* (3.19) *with right-hand side* (3.20) *has a unique solution $u^\kappa \in \mathcal{W}_\kappa(\Pi^\circ)$, which satisfies the estimate*

$$\|u^\kappa; \mathcal{W}_\kappa(\Pi^\circ)\| \leq c_\kappa \|f^\kappa; \mathcal{W}_{-\kappa}(\Pi^\circ)^*\|$$

and coincides with the (unique) solution $u^\flat \in \mathcal{H}(\Pi^\circ)$ of the problem (3.17) *with the right-hand side $f^\flat = f^\kappa \in \mathcal{H}(\Pi^\circ)^*$.*

Proof. For $u^\kappa \in \mathcal{W}_\kappa(\Pi^\circ)$ and $v^\kappa \in \mathcal{W}_{-\kappa}(\Pi^\circ)$, we set $\mathbf{u}^\kappa = \mathcal{E}_{-\kappa} u^\kappa$, $\mathbf{v} = \mathcal{E}_\kappa v^\kappa \in \mathcal{H}(\Pi^\circ)$. Then, we rewrite (3.19) as follows:

$$\begin{aligned} & a^\circ(\mathcal{E}_\kappa \mathbf{u}^\kappa, \mathcal{E}_{-\kappa}\mathbf{v}^\kappa; \Pi^\circ) - \lambda(M^\circ \mathbf{u}^\kappa, \mathbf{v}^\kappa)_{\Pi^\circ} \\ & + \sum_{k=1}^{J} (\psi_{(k)}, \mathbf{v}^\kappa)_{\Pi^\circ}(\mathbf{u}^\kappa, \psi_{(k)})_{\Pi^\circ} = \mathbf{f}^\kappa(\mathbf{v}^\kappa) := f^\kappa(\mathcal{E}_{-\kappa}\mathbf{v}^\kappa) \quad \forall \, \mathbf{v}^\kappa \in \mathcal{H}(\Pi^\circ). \end{aligned} \tag{3.21}$$

A simple computation gives us

$$
\begin{aligned}
& r^\circ_\kappa(\mathbf{u}^\kappa, \mathbf{v}^\kappa) \\
:= & \big(AD(\nabla)(\mathcal{E}_\kappa \mathbf{u}^\kappa), D(\nabla)(\mathcal{E}_{-\kappa}\mathbf{v}^\kappa)\big)_{\Pi^\circ} - \big(AD(\nabla)\mathbf{u}^\kappa, D(\nabla)\mathbf{v}^\kappa\big)_{\Pi^\circ} \\
= & \big(AD(\nabla)\mathcal{E}_\kappa)\mathbf{u}^\kappa), (D(\nabla)\mathcal{E}_{-\kappa})\mathbf{v}^\kappa\big)_{\Pi^\circ} + \big(A(D(\nabla)\mathcal{E}_\kappa)\mathbf{u}^\kappa, \mathcal{E}_{-\kappa} D(\nabla)\mathbf{v}^\kappa\big)_{\Pi^\circ} \\
& + \big(A\mathcal{E}_\kappa D(\nabla)\mathbf{u}^\kappa, (D(\nabla)\mathcal{E}_{-\kappa})\mathbf{v}^\kappa\big)_{\Pi^\circ}.
\end{aligned}
$$

The inequalities

$$\Big|\frac{\partial \mathcal{E}_{\pm\kappa}}{\partial x_j}(x)\Big| \le \kappa \mathcal{E}_{\pm\kappa}(x), \quad j = 1, \dots, d,$$

hold for the weight $\mathcal{E}_\kappa$, (3.18), and therefore

$$|r^\circ_\kappa(\mathbf{u}^\kappa, \mathbf{v}^\kappa)| \le c\kappa \|\mathbf{u}^\kappa; \mathcal{H}(\Pi^\circ)\| \, \|\mathbf{v}^\kappa; \mathcal{H}(\Pi^\circ)\|.$$

In other words, the operators of the problems (3.21) and (3.17) differ by an operator $\mathcal{H}(\Pi^\circ) \to \mathcal{H}(\Pi^\circ)^*$ with a small norm $O(\kappa)$ as $\kappa \to 0$.

Thus, in view of Proposition 3.3, problem (3.21) is also uniquely solvable[2] and therefore problem (3.19) has a solution $u^\kappa \in \mathcal{W}_\kappa(\Pi^{=}0)$ which is a solution of the problem (3.17) as well, because $\mathcal{W}_\kappa(\Pi^\circ) \subset \mathcal{H}(\Pi^\circ) \subset \mathcal{W}_{-\kappa}(\Pi^\circ)$. Hence, the claimed coincidence of the solutions follows from the uniqueness statement in Proposition 3.3. ☐

Note that a solution $u^\circ \in \ker \mathcal{B}^\circ(\lambda)$ of the problem (1.18) satisfies problem (3.17) with the right-hand side

$$f^\flat(v^\flat) = \sum_{k=1}^{K} (\psi_{(k)}, v^\flat)_{\Pi^\circ} (u^\circ, \psi_{(k)})_{\Pi^\circ},$$

which has a compact support and hence satisfies (3.20). This observation yields the following

Corollary 3.5. *If κ is as in Theorem 3.4, then there holds the inclusion $\ker \mathcal{B}^\circ(\lambda) \subset \mathcal{W}_\kappa(\Pi^\circ)$.*

3.2. **Weyl sequences.** In this section, the inclusion

$$\sigma \cup \sigma^\circ_{\rm di} \subset \sigma^\sharp_{\rm ess} \tag{3.22}$$

will be verified in a standard way, namely, by constructing singular sequences for the operator $\mathcal{A}^\sharp$ at a point $\mu \in \Sigma \cup \Sigma^\circ_{\rm di}$, see, e.g., [3, §1 Ch. 9], [39, VII.12].

Let $\lambda \in \sigma$ and $\mu = (1+\lambda)^{-1} \in \Sigma$. We consider the Floquet wave $w(x) = e^{i\eta^\top \alpha} U(x-\alpha)$ for $x \in \varpi(\alpha)$ and $\alpha \in \mathbb{Z}^d$, see (2.8), (2.9), and obtain for any $J \in \mathbb{N}$,

$$\|X_J w; \mathcal{H}(\Pi)\|^2 \ge (MX_J w, X_J w)_\Pi \ge (2J-1)^d. \tag{3.23}$$

Here we took into account that the number of the cells (1.3) on which $X_j = 1$ is exactly $(2J-1)^d$. Furthermore, since w satisfies the purely periodic problem in Π, we observe that, first,

$$a(w, X_J v; \Pi) = \lambda (Mw, X_J v)_\Pi \quad \forall\, v \in \mathcal{H}(\Pi),$$

[2]This and the further arguments in this proof are the very reason for inserting the sum on the left-hand side of (3.19) into the integral identity (3.1).

and, second, by (1.15), there holds

$$\|\mathcal{A}^\circ(X_J w) - \mu X_J w; \mathcal{H}(\Pi)^*\| = \sup \big|\langle \mathcal{A}^\circ(X_J w) - \mu X_J w, v\rangle\big|$$
$$= (1+\lambda)^{-1} \sup \big| a(X_J w, v; \Pi) - \lambda (M X_J w, v)_\Pi \big|$$
$$= (1+\lambda)^{-1} \sup \Big| \big(A(D(\nabla) X_J) w, D(\nabla) v\big)_\Pi$$
$$- \big(AD(\nabla) w, (D(\nabla) X_J) v\big)_\Pi \le C J^{d-1}. \tag{3.24}$$

Here, the supremum is computed over the unit ball of $\mathcal{H}(\Pi)$. To get the inequality at the end of (3.24) we used the fact that $\nabla X_J \neq 0$ only in the set $\Pi_J \setminus \Pi_{J-1}$ which contains $(2J+1)^d - (2J-1)^d = O(2d(2J)^{d-1})$ cells, by (3.4), and that $|D(\nabla) X_J(x)| \le c_X$ uniformly with respect to $J \in \mathbb{N}$.

The entries of the Weyl sequence are now defined as

$$W^J(x) = \|X_J w; \mathcal{H}(\Pi)\|^{-1} X_J^{\alpha(J)}(x) w(x - \alpha(J)), \tag{3.25}$$

where $X_J^\alpha(x) := X_J(x-\alpha)$ for $\alpha \in \mathbb{Z}^d$ and the shift vectors $\alpha(J) \in \mathbb{Z}^d$ are chosen such that

$$\Pi \cap \operatorname{supp} X_J^{\alpha(J)} = \Pi^\sharp \cap \operatorname{supp} X_J^{\alpha(J)}.$$

Thus, the support of (3.25) belongs to the set

$$\overline{\Pi} \setminus \bigcup_{k \in \mathbb{N}} \varpi(\alpha^k)$$

which is nothing but the part of the perturbed medium (1.19) which coincides with the original medium. The choice of the shift vectors is possible because the distance of adjacent foreign cells increases unboundedly due to the assumption (1.24). The condition

$$\operatorname{supp} W^J \cap \operatorname{supp} W^K = \varnothing \quad \text{for } J \neq K \tag{3.26}$$

can be satisfied for the same reason.

Formulas (3.25) and (3.26) readily imply the properties

1°. $\|W^J; \mathcal{H}(\Pi^\sharp)\| = 1$ for all $J \in \mathbb{N}$,

2°. $W^J \rightharpoonup 0$ weakly in $\mathcal{H}(\Pi^\sharp)$ as $J \to +\infty$

The third property, which is needed in order to make $\{W^J\}_{J=1}^\infty$ into a Weyl sequence, namely

3°. $\|\mathcal{A}^\sharp W^J - \mu W^J; \mathcal{H}(\Pi^\sharp)\| = 0$ as $J \to +\infty$,

is a consequence of (3.23)–(3.25).

The proof of the inclusion $\Sigma_{\rm di} \subset \Sigma_{\rm ess}^\sharp$ is much simpler because of the exponential decay of the eigenvectors of the problem (1.18), established in Corollary 3.5. Indeed, we take a function $w \in \ker \mathcal{B}^\circ(\lambda)$ with $\lambda \in \Sigma_d^\circ$, $(Mw, w)_{\Pi^\circ} = 1$, and obtain

$$\|X_J w; \mathcal{H}(\Pi^\circ)\|^2 \ge \frac{1}{2} - \|(1 - X_J) w; \mathcal{H}(\Pi^\circ)\|^2 \ge \frac{1}{2} - c e^{-2\kappa J},$$
$$\|\mathcal{A}^\circ(X_J w) - \mu X_J w; \mathcal{H}(\Pi^\circ)^*\|^2$$
$$= (1+\lambda)^{-1} \sup \big| a^\circ(X_J w, v; \Pi^\circ) - \lambda (M^\circ X_J w, v)_{\Pi^\circ} \big|$$
$$= (1+\lambda)^{-1} \sup \Big| \big(A(D(\nabla) X_J) w, D(\nabla) v\big)_{\Pi^\circ} - \big(AD(\nabla) w, (D(\nabla) X_J) v\big)_{\Pi^\circ}$$
$$\le C e^{-\kappa J}, \tag{3.27}$$

where the supremum is taken over functions v belonging to the unit sphere of $\mathcal{H}(\Pi^\circ)$. We define the vector function W^J as in (3.25), by changing Π into Π° and $\alpha(J)$ into α^p, where p is chosen such that $L_p \geq J$ and (3.26) holds, too. Now the properties 1°-3° of a Weyl sequence follow from (3.25) and (3.27), and the inclusion (3.22) is thus proven.

3.3. **Parametrix.** We fix the spectral parameter

$$\lambda \in \mathbb{R} \setminus (\sigma \cup \sigma^\circ_{\text{di}}) \tag{3.28}$$

and proceed to construct the right parametrix

$$\mathcal{R}^\sharp(\lambda) : \mathcal{H}(\Pi^\sharp)^* \to \mathcal{H}(\Pi^\sharp) \tag{3.29}$$

for the operator $\mathcal{B}^\sharp(\lambda) : \mathcal{H}(\Pi^\sharp) \to \mathcal{H}(\Pi^\sharp)^*$ of the inhomogeneous problem (1.22), i.e.

$$a^\sharp(u^\sharp, v^\sharp); \Pi^\sharp) - \lambda(M^\sharp u^\sharp, v^\sharp)_{\Pi^\sharp} = f^\sharp(v^\sharp) \quad \forall v^\sharp \in \mathcal{H}(\Pi^\sharp).$$

Accordingly, we will prove that the mapping

$$\mathcal{B}^\sharp(\lambda)\mathcal{R}^\sharp(\lambda) - \text{Id} : \mathcal{H}(\Pi^\sharp)^* \to \mathcal{H}(\Pi^\sharp)^* \tag{3.30}$$

is compact, since in view of the self-adjointness of $\mathcal{B}^\sharp(\lambda)$, this implies that the operator $\mathcal{B}^\sharp(\lambda)$ is Fredholm (cf. the explanation on the left parametrix after Lemma 3.2) and thus

$$\mu = (1+\lambda)^{-1} \notin \sigma^\sharp.$$

This, together with the inclusion (3.22), complete the proof of Theorem 2.3.

The proof will be given in several steps. In the step a) we divide the domain $\Pi^\sharp$ into two parts, one inside the box $\square_J$ (notation in (3.5)) and one outside it, for some large enough J. We apply the result of Section 3.1 to treat a given right-hand side $f^\sharp$ in the bounded subdomain $\square_J$. In the step b) we use a cut-off function to eliminate the right-hand side $f^\sharp$ near all foreign cells and solve the problem in a purely periodic domain. In the step c) we compensate the discrepancies caused by the previous approximations by solving an infinite family of problems in Π°. To show that all discrepancies only give rise to a compact operator we use the exponential decay of the solutions in Π° as explained in Section 3.1, the assumption on the sparse distribution of the foreign cells, and the choice of a large enough number J as a technical tool.

a). We fix an arbitrary natural number $m > 1$ and, by (1.24), assume that the number $J \in \mathbb{N}$ is large enough so that $L_p \geq m$ for any cell $\varpi^\bullet(\alpha^p) \subset \Pi^\sharp \setminus \square_J$, cf. (1.19) and (1.23). Let us denote by $\alpha^1, \dots, \alpha^{k^\circledcirc - 1}$ the indices of the cells $\varpi^\bullet(\alpha^k)$ which are contained in $\Pi^\sharp \cap \square_J$. For a moment, we regard Π as a J-periodic domain in d directions, so that the side lengths of the periodicity cells (call them J-cells) are equal to J. Let us define another new domain

$$\Pi^\circledcirc = (\Pi^\sharp \cap \square_J) \cup (\Pi \setminus \square_J),$$

which is obtained from Π by changing only one J-cell, namely $\Pi \cap \square_J$, by another one, $\Pi^\sharp \cap \square_J$. In the subdomain $\Pi \cap \square_J$ all J-cells remain unaltered. Up to a rescaling, the domain $\Pi^\circledcirc$ has the same geometric properties as Π° and in particular the results of Section 3.1 can be applied. More precisely, we now fix a functional $f^\sharp \in \mathcal{H}(\Pi^\sharp)^*$ and define the functional

$$f^\circledcirc \in \mathcal{H}(\Pi^\circledcirc)^* , \quad f^\circledcirc(v^\circledcirc) = f^\sharp(X_J v^\circledcirc) \tag{3.31}$$

so that $\operatorname{supp} f^\circledcirc \subset \overline{\Pi^\sharp} \cap \square_J$ (i.e., $f^\circledcirc$ vanishes on functions with support outside this subdomain). Next, we consider the analogue of the problem (3.1), where all quantities with the superindex $\circ$ are replaced by those with index $\circledcirc$ and where (3.31) is posed as the right-hand side. If the corresponding problem operator is denoted by $\mathcal{B}^\circledcirc : \mathcal{H}(\Pi^\circledcirc) \to \mathcal{H}(\Pi^\circledcirc)^*$, then the results of Section 3.1 yield a parametrix $\mathcal{R}^\circledcirc(\lambda) : \mathcal{H}(\Pi^\circledcirc)^* \to \mathcal{H}(\Pi^\circledcirc)$. We set

$$u^1 = X_{J+1}\mathcal{R}^\circledcirc(\lambda) f^\circledcirc \in \mathcal{H}(\Pi^\sharp). \tag{3.32}$$

b). To treat $f^\sharp$ outside $\square_J$, we define

$$f^1(v^\sharp) = f^\sharp((1 - X_J)v^\sharp) \tag{3.33}$$

and consider the purely periodic problem (with problem operator $\mathcal{B}^\Pi(\lambda) : \mathcal{H}(\Pi) \to \mathcal{H}(\Pi)^*$)

$$a(u^\Pi, v^\Pi; \Pi) - \lambda(Mu^\Pi, v^\Pi)_\Pi = f^\Pi(v^\Pi) \quad \forall\, v^\Pi \in \mathcal{H}(\Pi), \tag{3.34}$$

where

$$f^\Pi(v^\Pi) = f^1(\mathcal{X}_2 v^\Pi) \ \text{ and } \ \mathcal{X}_q(x) = \prod_{k \geq k^\circledcirc} \big(1 - X_q(x - \alpha^k)\big), \quad q = 1, 2.$$

We have $\mathcal{X}_2 = 0$ on all foreign cells $\varpi^\bullet(\alpha^k) \subset \Pi^\sharp \setminus \square_J$, hence, $f^\Pi \in \mathcal{H}(\Pi)^*$ and moreover,

$$\begin{aligned}
|f^\Pi(v)| &\leq \|f^1; \mathcal{H}(\Pi^\sharp)^*\|\, \|\mathcal{X}_2 v; \mathcal{H}(\Pi^\sharp)\| \\
&\leq c\|f^1; \mathcal{H}(\Pi^\sharp)^*\| \big(\|v; \mathcal{H}(\Pi)\| + \|v\nabla\mathcal{X}_2; L^2(\Pi)\|\big) \\
&\leq c\|f^1; \mathcal{H}(\Pi^\sharp)^*\|\, \|v; \mathcal{H}(\Pi)\| \\
&\Rightarrow \|f^\Pi; \mathcal{H}(\Pi)^*\| \leq \|f^\sharp; \mathcal{H}(\Pi^\sharp)^*\|.
\end{aligned} \tag{3.35}$$

Since $\lambda \notin \sigma$ by the assumption (3.28), the problem (3.34) has a unique solution u^Π satisfying the estimate

$$\|u^\Pi; \mathcal{H}(\Pi)\| \leq c\|f^\Pi; \mathcal{H}(\Pi)^*\| \leq C\|f^\sharp; \mathcal{H}(\Pi^\sharp)^*\|. \tag{3.36}$$

We set

$$u^2 = (1 - X_{J-1})\mathcal{X}_1 u^\Pi = (1 - X_{J-1})\mathcal{X}_1(\mathcal{B}^\Pi)^{-1} f^\Pi \tag{3.37}$$

and observe that analogously to (3.35),

$$\|u^2; \mathcal{H}(\Pi^\sharp)\| \leq c\|u^\Pi; \mathcal{H}(\Pi)\| \leq C\|f^\sharp; \mathcal{H}(\Pi^\sharp)^*\|.$$

c). To compensate the discrepancy left by (3.37) we set

$$\begin{aligned}
f^X(v) &= -\big(A(D(\nabla)X_{J-1})u^\Pi, D(\nabla)v\big)_\Pi + \big(AD(\nabla)u^\Pi, (D(\nabla)X_{J-1})v\big)_\Pi, \\
f_{(k)}(v) &= -f^1(X_2^k v) + \big(A(D(\nabla)X_1^k)u^\Pi, D(\nabla)v\big)_\Pi \\
&\quad -\big(AD(\nabla)u^\Pi, (D(\nabla)X_1^k)v\big)_\Pi
\end{aligned} \tag{3.38}$$

where we denote for $q = 1, 2$,

$$X_q^k = X_q \circ \tau^k \quad \text{with} \quad \tau^k(x) := x - \alpha^k \quad \forall\, x \in \mathbb{R}^d,\ k \in \mathbb{N}, \tag{3.39}$$

and X_q is as in (3.4). Observe that the mapping $f^\sharp \mapsto f^X \in \mathcal{H}(\Pi^\sharp)^*$ is compact, for the same reasons as in (3.11).

For every $k \in \mathbb{N}$, the functional (3.38) has a compact support in a neighborhood of the foreign cell $\varpi^\bullet(\alpha^k)$, and the shifted functional

$$f^\circ_{(k)}(v^\circ) = f_{(k)}(v^\circ_{(k)}), \quad v^\circ_{(k)} = v^\circ \circ \tau^k \tag{3.40}$$

belongs to $\mathcal{H}(\Pi^\circ)^*$ and $\mathcal{W}_{-\kappa}(\Pi^\circ)^*$ for all κ. We recall that due to (3.28), $\lambda \notin \sigma^\circ = \sigma \cup \sigma^\circ_{\mathrm{di}}$. Using Theorem 3.4 we thus obtain for all $k \in \mathbb{N}$, for any $\kappa \in (0, \kappa^0)$ a solution $u^\circ_{(k)} \in \mathcal{W}_\kappa(\Pi^\circ)^*$ of the problem

$$a^\circ(u^\circ_{(k)}, v^\kappa; \Pi^\circ) \,-\, \lambda(M^\circ u^\circ_{(k)}, v^\kappa)_{\Pi^\circ} = f^\circ_{(k)}(v^\kappa) \quad \forall\, v^\kappa \in \mathcal{W}_\kappa(\Pi^\circ), \tag{3.41}$$

such that

$$\|u^\circ_{(k)}; \mathcal{W}_\kappa(\Pi^\circ)\| \le c_0 \|f^\circ_{(k)}; \mathcal{W}_{-\kappa}(\Pi^\circ)^*\| \le c_0' \|f^\circ_{(k)}; \mathcal{H}(\Pi^\circ)^*\|. \tag{3.42}$$

Notice that the functions $\psi_{(k)}$ of the equation (3.19) do not appear in (3.41), since we are assuming that λ is not an eigenvalue of the problem (1.18) and thus $\dim \ker \mathcal{B}^\circ(\lambda) = 0$, see (3.28), the choice of the functions $\psi_{(k)}$ and the discussion above (3.16).

Finally, we set

$$u^3 = \sum_{k \ge k^\circledcirc} \chi_k\, u^\circ_{(k)} \circ \tau^k, \tag{3.43}$$

where we define one more family of cut-off-functions by $\chi_k(x) = X_{L_k/2} \circ \tau^k(x) = X_{L_k/2}(x - \alpha^k)$, $k \in \mathbb{N}$, so that the supports of the terms in (3.43) are mutually disjoint by the choice of the numbers L_k in (1.23).

We need to show that the map

$$\mathcal{H}(\Pi^\sharp)^* \ni f^\sharp \mapsto \sum_{k \ge k^\circledcirc} \widetilde{f}^3_{(k)} \in \mathcal{H}(\Pi^\sharp)^*, \tag{3.44}$$

where

$$\begin{aligned} \widetilde{f}^3_{(k)}(v) \,=\, &-\big(A(D(\nabla)\chi_k) u^\circ_{(k)} \circ \tau^k, D(\nabla) v\big)_{\Pi^\sharp} \\ &+\big(A D(\nabla) u^\circ_{(k)} \circ \tau^k, (D(\nabla)\chi_k) v\big)_{\Pi^\sharp}, \end{aligned} \tag{3.45}$$

is compact (since we will see that this sum appears in the discrepancy caused by the series (3.43)). By the same argument as after (3.10) one can see that a single mapping $f^\sharp \mapsto \widetilde{f}^3_{(k)}(v)$ is compact, but this is not enough to conclude the same property for the whole infinite sum of them. However, we take into account the weight (3.18), which is of order $O(e^{\kappa L_k/2})$ on the set

$$(\tau^k)^{-1}(S_k), \quad \text{where } S_k = \operatorname{supp}\big(|D(\nabla)\chi_k|\big),$$

see (3.4) and the definition of χ_k just above. Thus, we obtain for every $v \in \mathcal{H}(\Pi^\sharp)$ with $\|v; \mathcal{H}(\Pi^\sharp)\| \le 1$ and for every $k \ge k^\circledcirc$, by (3.45),

$$|\widetilde{f}^3_{(k)}(v)| \le c\|u^\circ_{(k)} \circ \tau^k; H^1(S_k)\| = c\big\|u^\circ_{(k)}; H^1\big((\tau^k)^{-1}(S_k)\big)\big\|$$

$$(3.46)\quad = ce^{-\kappa L_k/2}\|e^{\kappa L_k/2}u^\circ_{(k)};H^1\big((\tau^k)^{-1}(S_k)\big)\| \le c'e^{-\kappa L_k/2}\|u^\circ_{(k)};\mathcal{W}_\kappa(\Pi^\circ)\|.$$

By (3.38), (3.33), (3.36), there also holds $\|f^\circ_{(k)};\mathcal{H}(\Pi^\circ)^*\| \le c\|f^\sharp;\mathcal{H}(\Pi^\sharp)^*\|$ for all k so that we get by (3.46), (3.42) and the disjointness of the supports of the functionals $f^\circ_{(k)}$

$$\sum_{k\ge k^\circledcirc} e^{\kappa L_k}\|\widetilde{f}^3_{(k)};\mathcal{H}(\Pi^\sharp)^*\|^2 \le c\sum_{k\ge k^\circledcirc}\|u^\circ_{(k)};\mathcal{W}_\kappa(\Pi^\circ)\|^2$$

$$(3.47)\qquad \le c'\sum_{k\ge k^\circledcirc}\|f^\circ_{(k)};\mathcal{H}(\Pi^\sharp)^*\|^2 \le c_\kappa\|f^\sharp;\mathcal{H}(\Pi^\sharp)^*\|^2.$$

We can now conclude that the operator (3.44) is compact, since it can be presented for any $\varepsilon > 0$ as the sum $F^\varepsilon(f^\sharp) + F^\varepsilon_{\rm comp}(f^\sharp)$, where $F^\varepsilon_{\rm comp}$ is compact and F^ε has norm less than ε. Indeed, by the key assumption (1.24) we can choose $k_\varepsilon \ge k^\circledcirc$ large enough such that $c_\kappa e^{-\kappa L_j} \le \varepsilon^2$ for all $j \ge k_\varepsilon$, where the constant c_κ is as in (3.47), and then define

$$F^\varepsilon(f^\sharp) = \sum_{k\ge k_\varepsilon}\widetilde{f}^3_{(k)}.$$

so that (3.47) and the above choice imply

$$\|F^\varepsilon(f^\sharp);\mathcal{H}(\Pi^\sharp)\|^2 \le \sum_{k\ge k_\varepsilon} c_\kappa^{-1}e^{\kappa L_j}\varepsilon^2\|\widetilde{f}^3_{(k)};\mathcal{H}(\Pi^\sharp)\|^2 \le \varepsilon^2\|f^\sharp;\mathcal{H}(\Pi^\sharp)^*\|^2.$$

The operator defined by the finite sum

$$F^\varepsilon_{\rm comp}(f^\sharp) = \sum_{k<k_\varepsilon}\widetilde{f}^3_{(k)}$$

is compact by what was said about single terms.

d). We now define the parametrix (3.29) by combining the expressions (3.32), (3.37) and (3.43):

$$(3.48)\qquad \mathcal{R}^\sharp(\lambda)f^\sharp = u^1 + u^2 - u^3.$$

To prove that (3.30) is indeed a compact operator, we fix $v^\sharp \in \mathcal{H}(\Pi^\sharp)$. First, let us employ (3.31), (3.32):

$$\begin{aligned}
&\mathcal{B}^\sharp(\lambda)u^1(v^\sharp)\\
&= \big(AD(\nabla)(X_{J+1}\mathcal{R}^\circledcirc f^\circledcirc), D(\nabla)v^\sharp\big)_{\Pi^\circledcirc} - \lambda\big(MX_{J+1}\mathcal{R}^\circledcirc f^\circledcirc, v^\sharp\big)_{\Pi^\circledcirc}\\
&= \big(AD(\nabla)\mathcal{R}^\circledcirc f^\circledcirc, D(\nabla)(X_{J+1}v^\sharp)\big)_{\Pi^\circledcirc} - \lambda\big(M\mathcal{R}^\circledcirc f^\circledcirc, X_{J+1}v^\sharp\big)_{\Pi^\circledcirc}
\end{aligned}$$

$$(3.49)\quad +\big(A(D(\nabla)X_{J+1})\mathcal{R}^\circledcirc f^\circledcirc, D(\nabla)v^\sharp\big)_{\Pi^\circledcirc} - \big(AD(\nabla)\mathcal{R}^\circledcirc f^\circledcirc, v^\sharp D(\nabla)X_{J+1}\big)_{\Pi^\circledcirc}.$$

Since $X_{J+1}X_J = X_J$, the penultimate row equals

$$\begin{aligned}
&\mathcal{B}^\circledcirc(\lambda)\mathcal{R}^\circledcirc(\lambda)f^\circledcirc(X_{J+1}v^\sharp) = f^\circledcirc(X_{J+1}v^\sharp) + \widetilde{\mathcal{K}}^1 f^\circledcirc(X_J v^\sharp)\\
&= f^\sharp(X_J v^\sharp) + \mathcal{K}^1 f^\sharp(v^\sharp),
\end{aligned}$$

where $\widetilde{\mathcal{K}}^1 : \mathcal{H}^\circledcirc(\Pi^\circledcirc)^* \to \mathcal{H}^\circledcirc(\Pi^\circledcirc)^*$ is a compact operator, and thus also the operator $\mathcal{K}^1 : \mathcal{H}^\sharp(\Pi^\sharp)^* \to \mathcal{H}^\sharp(\Pi^\sharp)^*$ defined by

$$(3.50)\qquad \mathcal{K}^1 f^\sharp(v^\sharp) = \widetilde{\mathcal{K}}^1 f^\circledcirc(X_J v^\sharp)$$

is compact. The last line of (3.49) is denoted by $\widetilde{f}^1(v^\sharp)$, and the map $f^\sharp \mapsto \widetilde{f}^1 \in \mathcal{H}(\Pi^\sharp)^*$ is compact, for the same reasons as in (3.11). We obtain

$$\mathcal{B}^\sharp(\lambda)u^1(v^\sharp) = f^\sharp(X_J v^\sharp) + \mathcal{K}^1 f^\sharp(v^\sharp) + \widetilde{f}^1(v^\sharp). \tag{3.51}$$

Next, we use (3.33), (3.34) , the identities $\mathcal{X}_1\mathcal{X}_2 = \mathcal{X}_2$ and $(1 - X_J)(1 - X_{J-1}) = 1 - X_J$, and $f^\Pi(v^\sharp) = f^\sharp(\mathcal{X}_2(1 - X_J)v^\sharp)$. The following argument is similar although much simpler than (3.49)–(3.51), since we can use the unique solution of (3.34) instead of the parametrix $\mathcal{R}^\circledcirc$:

$$\begin{aligned}
&\mathcal{B}^\sharp(\lambda)u^2(v^\sharp)\\
&= \Big(AD(\nabla)\big((1 - X_{J-1})\mathcal{X}_1(\mathcal{B}^\Pi)^{-1}f^\Pi\big), D(\nabla)v^\sharp\big)\Big)_\Pi\\
&\quad -\lambda\Big(M(1 - X_{J-1})\mathcal{X}_1(\mathcal{B}^\Pi)^{-1}f^\Pi), v^\sharp)\Big)_\Pi\\
&= f^\sharp(\mathcal{X}_2(1 - X_J)v^\sharp) + \Big(Au^\Pi D(\nabla)\big((1 - X_{J-1})\mathcal{X}_1\big), D(\nabla)v^\sharp)\Big)_{\Pi^\circledcirc}\\
&\quad -\Big(AD(\nabla)u^\Pi, v^\sharp D(\nabla)\big((1 - X_{J-1})\mathcal{X}_1\big)\Big)_{\Pi^\circledcirc}\\
&= f^\sharp(\mathcal{X}_2(1 - X_J)v^\sharp) - \big(A(D(\nabla)X_{J-1})u^\Pi, D(\nabla)v^\sharp\big)_\Pi\\
&\quad +\big(AD(\nabla)u^\Pi, (D(\nabla)X_{J-1})v^\sharp\big)_{\Pi},\\
&\quad + \sum_{k\geq k^\circledcirc} \big(A(D(\nabla)X_1^k)u^\Pi, D(\nabla)v^\sharp\big)_\Pi\\
&\quad -\big(AD(\nabla)u^\Pi, (D(\nabla)X_1^k)v^\sharp\big)_\Pi\\
&= f^\sharp(\mathcal{X}_2(1 - X_J)v^\sharp) + f^X(v^\sharp) + \sum_{k\geq k^\circledcirc} \Big(f_{(k)}(v^\sharp) + f^1(X_2^k v^\sharp)\Big),
\end{aligned}\tag{3.52}$$

where we at the end used the notation (3.38).

Finally, by (3.43),

$$\begin{aligned}
&\mathcal{B}^\sharp(\lambda)u^3(v^\sharp)\\
&= \sum_{k\geq k^\circledcirc} \bigg(\Big(AD(\nabla)(\chi_k u^\circ_{(k)} \circ \tau^k), D(\nabla)v^\sharp\Big)_{\Pi^\sharp} - \lambda\big(M\chi_k u^\circ_{(k)} \circ \tau^k, v^\sharp\big)_{\Pi^\sharp}\bigg)\\
&= \sum_{k\geq k^\circledcirc} \Big(\big(AD(\nabla)(u^\circ_{(k)} \circ \tau^k), D(\nabla)(\chi_k v^\sharp)\big)_{\Pi^\sharp} - \lambda\big(Mu^\circ_{(k)} \circ \tau^k, \chi_k v^\sharp\big)_{\Pi^\sharp}\\
&\quad +\big(A(D(\nabla)\chi_k)u^\circ_{(k)} \circ \tau^k, D(\nabla)v^\sharp\big)_{\Pi^\sharp}\\
&\quad -\big(AD(\nabla)u^\circ_{(k)} \circ \tau^k, (D(\nabla)\chi_k)v^\sharp\big)_{\Pi^\sharp}\Big).
\end{aligned}$$

Here, the penultimate line is by (3.41), (3.40) equal to

$$\sum_{k\geq k^\circledcirc} f_{(k)}(\chi_k v^\sharp),$$

and the terms on the last line are equal to $\widetilde{f}^3_{(k)}(v^\sharp)$, by (3.45). Due to the supports of the cut-off functions X_1^k and χ_k, see (3.38), (3.43) we have $f_{(k)}(\chi_k v^\sharp) = f_{(k)}(v^\sharp)$

for all k, hence,

$$-\mathcal{B}^\sharp(\lambda)u^3(v^\sharp) = -\sum_{k\geq k^\circledcirc}\Big(f_{(k)}(v^\sharp)+\widetilde{f}^3_{(k)}(v^\sharp)\Big). \tag{3.53}$$

Summing up (3.51), (3.52) and (3.53) (see (3.48)) yields for all $v^\sharp \in \mathcal{H}(\Pi^\sharp)$.

$$\mathcal{B}^\sharp(\lambda)\mathcal{R}^\sharp(\lambda)f^\sharp(v^\sharp) = f^\sharp(v^\sharp) + \mathcal{K}^1 f^\sharp(v^\sharp) + \widetilde{f}^1(v^\sharp) + f^X(v^\sharp) - \sum_{k\geq k^\circledcirc}\widetilde{f}^3_{(k)}(v^\sharp).$$

The right hand side forms, as desired, a compact perturbation of the identity mapping of $\mathcal{H}(\Pi^\sharp)^*$, by the remarks around (3.50), (3.51), (3.39), (3.45). □

4. Possible generalizations.

4.1. **Geometry.** One can generalize the results of the previous sections for example by considering, instead of a single cell $\varpi^\bullet$, several types of such cells $\varpi^\bullet_{(1)},\ldots,\varpi^\bullet_{(m)}$, which have the characteristics $A^\bullet_{(q)}$, $M^\bullet_{(q)}$ and $P^\bullet_{(q)}$, $1=1,\ldots,m$, and satisfy the sparseness assumption (1.24) with sequences $\{L_p^{(q)}\}_{p\in\mathbb{N}}$ for the cubes $\boxdot^p_{(q)} \supset \varpi^\bullet_{(q)}(\alpha^p_{(q)})$ of size $2L_p^{(q)}+1$, cf. (1.23). The essential spectrum of the problem (1.22) in the medium $\Pi^\sharp$ with the family $\big\{\varpi^\bullet_{(q)}(\alpha^p_{(q)}) \,:\, p\in\mathbb{N}, q=1,\ldots,m\big\}$ of inclusions is

$$\sigma^\sharp_{\rm ess} = \sigma \cup \bigcup_{q=1}^m \sigma^{\circ,(q)}_{\rm di}$$

where $\sigma^{\circ,(q)}_{\rm di}$ is the discrete spectrum of the problem (1.18) corresponding to the foreign cell $\varpi^\bullet_{(q)}$ instead of $\varpi^\bullet$, see Sections 1.3 and 2.2.

One may also consider sparsely distributed identical conglomerates of miscellaneous foreign inclusions. This can be done along the scheme which was explained in Remark 1.2 in the simple case of the duplication of neighbouring cells. The general case can be studied by using the coordinate dilation

$$x \mapsto (\tau_1^{-1}x_1,\ldots,\tau_d^{-1}x_d), \quad \tau=(\tau_1,\ldots,\tau_d)\in\mathbb{N}^d,$$

which puts the conglomerate inside one cell of size one. It is worth mentioning that such affine transforms of Cartesian coordinates preserve the linear elasticity equations in $d=2,3$, if one uses the Voigt-Mandel notation and introduces artificial, non-physical, displacements, strains and stresses. See, e.g. [20].

Our approach, with minor modifications, also applies to layer-like composites, where the space $\mathbb{R}^d = \cup_{\alpha\in\mathbb{Z}^d}\overline{\square(\alpha)}$ is replaced by the layer $\mathbb{L}_\boxminus$ paved with the cells

$$\varpi_\boxminus = \varpi(\alpha)\times\omega \subset \mathbb{R}^{d+d_\boxminus}$$

where ω is a bounded Lipschitz domain in the space $\mathbb{R}^{d_\boxminus}$ of dimension $d_\boxminus > 1$.

Our method also works for lattices different from the cubic one, (1.2), (1.1), for example, for the honeycomb lattice (cf. [19, 34]). One can find a detailed description of such lattices, e.g., in [44].

4.2. **Absolutely rigid inclusions in elasticity.** Let $\Pi \subset \mathbb{R}^3$ be a triply periodically perforated Euclidean space, namely, define the periodicity cells by $\varpi = \square \setminus \overline{\omega}$, where $\omega \neq \varnothing$ is a Lipschitz domain inside the cube, $\overline{\omega} \subset \square$. Then, define the domain Π as in (1.1)–(1.3). We wish to study the linear elasticity problem in Π with boundary conditions describing the contact with absolutely rigid bodies $\omega(\alpha)$, $\alpha \in \mathbb{Z}^3$. However, due to the topology of the situation, the usual Dirichlet condition (1.5) (corresponding to $P = \mathbb{I}_d$ in (1.12)) cannot be used now, since the forces acting on the surfaces $\partial\omega(\alpha)$ should be balanced by some weird, impossible non-physical activity.

Instead of the Dirichlet conditions, the following boundary conditions on the isolated surfaces are appropriate from the mechanical point of view,

$$u(x) = d(x)c^\alpha, \quad x \in \partial\omega(\alpha) \tag{4.1}$$

$$\int_{\partial\omega(\alpha)} d(x)^\top D(\nu(x))^\top A(x) D(\nabla) u(x)dx = 0 \in \mathbb{R}^6, \tag{4.2}$$

where c^α can be an arbitrary column in $\mathbb{R}^6$ and $d(x)$ is the following 3×6-matrix of rigid motions,

$$d(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & -2^{-1/2}x_3 & 2^{-1/2}x_2 \\ 0 & 1 & 0 & 2^{-1/2}x_3 & 0 & -2^{-1/2}x_1 \\ 0 & 0 & 1 & -2^{-1/2}x_2 & 2^{-1/2}x_1 & 0 \end{pmatrix};$$

compare with the structure of the matrix $D^\top$ in (1.25). According to (4.1), the displacement vector u is a rigid motion (a linear combination of the three translations and three rotations) of the rigid body $\omega(\alpha)$, while the six relations in (4.2) make the traction force $D(\nu)^\top A D(\nabla)u$ (see the Hooke law (1.27)) on the surface $\partial\omega(\alpha)$ self-balanced.

The variational formulation of the elasticity problem is posed in the space

$$\mathcal{H}(\Pi) = \{u \in H^1(\Pi)^3 \,:\, u\big|_{\partial\omega(\alpha)} \in \mathcal{D}^\alpha,\ \alpha \in \mathbb{Z}\}, \tag{4.3}$$

where

$$\mathcal{D}^\alpha = \{u \,:\, u(x) = d(x-\alpha)c^\alpha \text{ for some } c^\alpha \in \mathbb{R}^6,\ x \in \partial\omega(\alpha)\}.$$

Notice that the sequence $\{c^\alpha\}_{\alpha\in\mathbb{Z}^d}$ is not fixed a priori, but it is found by solving the whole problem. We also remark that the integral conditions (4.2) have been derived form the integral identity (1.14) by using the Green formula and the arbitrariness of c^α in (4.3).

Although the space (4.3) is not formally included in the scheme of Sections 2 and 3, the method can still clearly be applied to prove the above statements.

In dimension $d = 2$ we have

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 \end{pmatrix}, \quad d(x) = \begin{pmatrix} 1 & 0 & 2^{-1/2}x_2 \\ 0 & 1 & -2^{-1/2}x_1 \end{pmatrix}$$

and A is a symmetric, positive definite matrix of size 3×3, whose entries are real valued functions. In this case the Dirichlet boundary conditions make sense, since any part of the two-dimensional plane can be reached from outside it.

4.3. **Kirchhoff plates.** Let us consider a two-dimensional model of a thin elastic anistropic plate see [22, 40, 26, 10] and many others. This is a fourth order analogue of the (scalar) equation (1.4): in the operator (1.6), the real-valued function matrix A of size 3×3 is assumed symmetric and positive definite, but $D(\nabla)$ is replaced by the second order column operator

$$D(\nabla)^\top = \Big(\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \sqrt{2}\frac{\partial^2}{\partial x_1 \partial x_2}\Big).$$

Repeating the proofs in this case would require many changes (for example, the continuous weight functions (3.18) should be made differentiable), however, the scheme would work as a whole and it would lead to conclusions similar to Theorem 2.3. We refrain from formulating the exact results and instead only mention that in this elastic plate model, the Dirichlet (clamping) boundary conditions of the second order model are replaced by

$$u(x) = 0, \ \partial_\nu u(x) = 0$$

on the edges of the plate, i.e. for $x \in \partial\Pi$. The boundary conditions, which correspond to the Neumann or mixed conditions of the second order case, are much more complicated, see the monographs cited above.

4.4. **Piezoelectric media.** We set $d = 3$, $n = 4$, $N = 9$ and denote $D^{\mathsf{E}}(\nabla) = \nabla$ and (as in (1.25)) $D^{\mathsf{M}}(\nabla) = D(\nabla)$, and introduce the 9×4- and 9×9-matrices

$$D(\nabla) = \begin{pmatrix} D^{\mathsf{M}}(\nabla) & \mathbb{O}_{6\times 1} \\ \mathbb{O}_{3\times 3} & D^{\mathsf{E}}(\nabla) \end{pmatrix}, \quad A = \begin{pmatrix} A^{\mathsf{MM}} & -A^{\mathsf{ME}} \\ A^{\mathsf{EM}} & A^{\mathsf{EE}} \end{pmatrix}; \tag{4.4}$$

the superscripts M and E stand for "mechanical" and "electrical". Furthermore, A^{MM} and A^{EE} are the elastic and dielectric matrices, which are real, symmetric, positive definite, and of sizes 6×6 and 3×3, respectively. No restriction is posed on the real piezoelectric matrix $A^{\mathsf{ME}} = \big(A^{\mathsf{EM}}\big)^\top$, except that it is not the null matrix.

Although the matrix A is not symmetric, the spectrum of the piezoelectricity system (1.4) with appropriate boundary conditions (1.5) is contained in the set of non-negative real numbers, see for example [35, 31, 21] and others. This is a consequence of the specific structure of the diagonal matrix

$$M(x) = \varrho^{\mathsf{M}}(x) \mathrm{diag}\,\{1, 1, 1, 0\}$$

on the right-hand side of (1.4); here $\varrho^{\mathsf{M}} > 0$ is the material density, see Section 1.4, 2°. The vector function $u = (u_1^{\mathsf{M}}, u_2^{\mathsf{M}}, u_3^{\mathsf{M}}, u_4^{\mathsf{E}})^\top$ is composed of the displacement vector $u^{\mathsf{M}} = (u_1^{\mathsf{M}}, u_2^{\mathsf{M}}, u_3^{\mathsf{M}})^\top$ and the electric potential $u^{\mathsf{E}} = u_4^{\mathsf{E}}$.

The Neumann boundary condition

$$D(\nu(x))^\top A(x) D(\nabla) u(x) = 0, \quad x \in \partial\Pi,$$

means that the holes $\omega(\alpha)$ consist of vacuum, which is an insulator and corresponds to a traction-free boundary.

The Dirichlet conditions

$$u^{\mathsf{M}}(x) = 0, \ u^{\mathsf{E}}(x), \quad x \in \partial\Pi, \tag{4.5}$$

correspond to the ideal contact of the piezoelectric medium with an absolutely rigid conductor, but this setting has a clear physical sense only in the case $\mathbb{R}^d \setminus \overline{\Pi}$ is a connected set. If the domain is perforated by isolated voids, the conditions (4.5) must be reformulated in the same way as in Section 4.2 for both the mechanical and electric components, because the electric potential becomes constant at each isolated conductor surface $\partial\omega(\alpha)$; these constans may differ from each other for different α.

The piezoelectricity problem can be reduced to a study of a self-adjoint operator, see [35, 31, 21], but this operator contains a non- trivial integro-differential operator term, the definition of which is only implicit. Applying the theory of self-adjoint semibounded Hilbert space operators is still possible, but the calculations become quite troublesome, see the papers cited above. It is thus more convenient to deal directly with the operators generated by the integral identities. The key observation in doing so is that for matrices (4.4) we have

$$\begin{aligned}
&(AD(\nabla)u, D(\nabla)u)_\Pi = a(u,u;\Pi) + b(u,u;\Pi),\\
&a(u,u;\Pi) = \big(A^{\mathsf{MM}} D^{\mathsf{M}}(\nabla)u^{\mathsf{M}}, D^{\mathsf{M}}(\nabla)u^{\mathsf{M}}\big)_\Pi + \big(A^{\mathsf{EE}} D^{\mathsf{E}}(\nabla)u^{\mathsf{E}}, D^{\mathsf{E}}(\nabla)u^{\mathsf{E}}\big)_\Pi\\
&b(u,u;\Pi) = \big(A^{\mathsf{EM}} D^{\mathsf{M}}(\nabla)u^{\mathsf{M}}, D^{\mathsf{E}}(\nabla)u^{\mathsf{E}}\big)_\Pi - \big(A^{\mathsf{ME}} D^{\mathsf{E}}(\nabla)u^{\mathsf{E}}, D^{\mathsf{M}}(\nabla)u^{\mathsf{M}}\big)_\Pi.
\end{aligned}$$

Here, most importantly,

$$\begin{aligned}
&a(u,u;\Pi) \geq c_A \|D(\nabla)u; L^2(\Pi)\|^2\\
&\operatorname{Re} b(u,u;\Pi) = 0
\end{aligned}$$

so that the Lax-Milgram lemma can be applied. However, additional considerations are needed for the investigation of the spectra of piezoelectric media with either localized or sparsely placed defects, and we leave this topic to a planned forthcoming papers by the authors.

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Manuscript received March 10 2019
revised July 9 2019

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